Fall 2021 – Math 328K – 55385

Homework 1

due Tuesday, September 7, 14:00

Problem 1. Show that, for all sets X, Y and Z,

1. $X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$,
2. $X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$,
3. $X \setminus (X \setminus Y) = X \cap Y$,
4. $X \setminus (Y \cup Z) = (X \setminus Y) \cap (X \setminus Z) = (X \setminus Y) \setminus Z$,
5. $X \setminus (Y \cap Z) = (X \setminus Y) \cup (X \setminus Z)$.

Problem 2. Using induction, show that for all $n \in \mathbb{N}$ we have

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

Problem 3. Show that $2^n \geq n^2$ for every integer $n \geq 4$.

Problem 4. Let R be an ordered ring with $R_+ \subset R$ its set of positive elements. Derive from the axioms that

a) $(-a) \cdot (-b) = ab$ for all $a, b \in R$,

b) $a \cdot a > 0$ for all $a \in R$ unless $a = 0$,

c) if $a, b, c \in R$ with $a > b$ and $b > c$, then $a > c$.

Problem 5. Let R be a ring and $r \in R$. Show that

$$(r-1) \sum_{k=0}^{n} r^k = r^{n+1} - 1.$$

In particular, if R is $\mathbb{Q}$ or $\mathbb{R}$ or $\mathbb{C}$ and $r \neq 1$, then

$$\sum_{k=0}^{n} r^k = \frac{r^{n+1} - 1}{r - 1}.$$