Homework 3

due Tuesday, September 21, 14:00

Problem 1. How many integer solutions $x, y \in \mathbb{Z}$ do the following equations have? If there is at least one solution, find an example.

a) $60x + 18y = 97$

b) $37x + 1000010000100001y = 0$

c) $14541x + 1367631y = 13566531$

Problem 2. Let a, b be positive integers such that $a^2 \mid b^2$. Show that $a \mid b$.

Problem 3. Denote by p_n be the n–th prime ($p_1 = 2, p_2 = 3, p_3 = 5, \dots$). Show that $p_n \leq 2^{2^{n-1}}$ for every $n \in \mathbb{N}$. Conclude that there are at least $n+1$ primes less than 2^{2^n} for every $n \in \mathbb{N}$.

Hint: Look at Euclid's proof of the infinitude of primes.

Problem 4. Let $p \neq q$ be prime numbers and a be an integer such that $p \mid a$ and $q \mid a$. Show that $pq \mid a$. Find a counterexample if either p or q is not prime.

Problem 5. Let $a, b, c \in \mathbb{Z}$ and write $d = (a, b)$. We proved in class that the equation $ax + by = c$ has an integer solution if and only if $d \mid c$.

Now assume that $d \mid c$, and let s, t be a pair of Bezout coefficients for a and b, so that $as + bt = d$. Show that the set of all solutions (x, y) of $ax + by = c$ is

$$\left\{ \left(\frac{cs - nb}{d}, \frac{ct + na}{d} \right) \mid n \in \mathbb{Z} \right\}.$$