Homework 4

due Tuesday, September 28, 14:00

Problem 1. Which pairs $(a,b) \in \mathbb{Z}_+ \times \mathbb{Z}_+$ have $\gcd(a,b) = 18$ and $\operatorname{lcm}(a,b) = 540$?

Problem 2. Show that there are no integer solutions to $x^2 + y^2 = 1000003$.

Problem 3. Find all integer solutions (x,y,z) to the equation

$$2x + 3y + 4z = 5.$$

Which ones have $x, y, z \geq 0$?

Problem 4. To understand the multiplication operation on $\mathbb{Z}/m\mathbb{Z}$ better, it can be helpful to compile a multiplication table, that is to list the product $x \cdot y$ for all pairs $x, y \in \mathbb{Z}/m\mathbb{Z}$. To do this, we should fix a single representative for every congruence class, for example by using the least non–negative representatives. For example, the multiplication table for $\mathbb{Z}/3\mathbb{Z}$ would look like this (for example $[2]_3 \cdot [2]_3 = [4]_3 = [1]_3$):

	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

Create a multiplication table for $\mathbb{Z}/4\mathbb{Z}$, $\mathbb{Z}/5\mathbb{Z}$ and $\mathbb{Z}/6\mathbb{Z}$. Do you see interesting patterns?

Problem 5. Let $m \in \mathbb{Z}_+$ be composite. Show that $\mathbb{Z}/m\mathbb{Z}$ is not an integral domain.

Problem 6. Compute the following expressions. Give the answer as a least non–negative residue. You can (and should!) do this without a calculator, and write down intermediate steps.

a) $2^6 \bmod 11$

b) $2^{12} \bmod 11$

c) $5^{1030} \bmod 3$

d) $78^3 \bmod 3$

e) $(1! + 2! + \cdots + 10!) \bmod 5$

f) $(1! + 2! + \cdots + 100!) \bmod 15$