Fall 2021 – Math 328K – 55385

Homework 8

due Thursday, October 28, 14:00

Problem 1. What are the last four decimal digits of the number 11^{15999}?

Problem 2. Show the following facts about Euler's ϕ–function:

a) $\phi(n)$ is even for every $n \geq 3$,

b) $\phi(n^k) = n^{k-1}\phi(n)$ for all $n, k \in \mathbb{N}$,

c) $\phi(n) \geq \sqrt{n}$ for all $n \in \mathbb{N} \setminus \{2, 6\}$,

d) If $m \mid n$, then $\phi(m) \mid \phi(n)$.

Problem 3. Let $n = p_1 \cdots p_k$ be a product of distinct odd primes and let $x \in \mathbb{Z}/n\mathbb{Z}$. Show that

$$x^{\phi(n)+1} = x.$$

Problem 4. Let $m \in \mathbb{N}$. The goal of this problem is to find all integers which are congruent modulo m to their own square. In other words, we want to find all solutions of the equation $x^2 - x = 0$ in $\mathbb{Z}/m\mathbb{Z}$.

a) Show that, if m is prime, then the only solutions are $[0]$ and $[1]$.

b) Show that, if m is a prime power, then the only solutions are still $[0]$ and $[1]$.

c) For general m, let $m = p_1^{i_1} \cdots p_k^{i_k}$ be the prime–power decomposition of m with $p_1 < \cdots < p_k$ prime and $i_1, \ldots, i_k \in \mathbb{N}$. Show there are 2^k different solutions of the equation $x^2 - x = 0$ in $\mathbb{Z}/m\mathbb{Z}$, and that these are given by

$$\sum_{j=1}^{k} \delta_j \left(\frac{m}{p_j^{i_j}}\right)^{p_j^{i_j} - p_j^{i_j - 1}} \mod m$$

for every tuple $(\delta_1, \ldots, \delta_k) \in \{0, 1\}^k$.

Hint: For the last part, use Euler's theorem and the Chinese remainder theorem.