Homework 9

due Thursday, November 4, 14:00

Problem 1. Which positive integers m have exactly 3/4/5 positive divisors?

Problem 2. Let

$$\mathcal{F} = \{\text{functions } f\colon \mathbb{Z}_+ \to \mathbb{R}\}$$

be the set of all real valued arithmetic functions. We want to define two binary operations on $\mathcal{F}$, an addition $+$ and a multiplication $\star$. The operation $+$ is the pointwise addition of functions, defined by

$$(f+g)(n) = f(n) + g(n),$$

for any $f, g \in \mathcal{F}$ and $n \in \mathbb{Z}_+$, and $f \star g$ is the *Dirichlet product*

$$(f \star g)(n) = \sum_{d|n} f(d)g(n/d).$$

Prove the following statements, for $f, g \in \mathcal{F}$.

a) $\mathcal{F}$ with the addition $+$ and multiplication $\star$ is a commutative ring.

Hint: as multiplicative identity, take the arithmetic function ι defined by $\iota(1) = 1$ and $\iota(n) = 0$ for all $n > 1$. You should convince yourself that all ring axioms are true, but it is enough if you write down commutativity, and associativity of the Dirichlet product, and prove that ι is the neutral element.

b) $f \in \mathcal{F}^\times$ if and only if $f(1) \neq 0$.

Hint: find a recursive formula for f^{-1}.

c) If f and g are multiplicative, then $f \star g$ is multiplicative.

d) If f is multiplicative and invertible, then f^{-1} is multiplicative.

This is a bit more challenging, and not strictly needed for the rest of the proof. Maybe skip it first, and come back to it when you're done with the rest.

e) Let $\nu \in \mathcal{F}$ be the constant function $\nu(n) = 1$. Then the inverse of ν is the Möbius function μ (that is, the multiplicative function defined by $\mu(p) = -1$ and $\mu(p^k) = 0$ for every prime number p and $k \geq 2$).

f) Let f be an arithmetic function and F its summatory function. Then $F = f \star \nu$ and $f = F \star \mu$. This is the *Möbius inversion formula.* If F is multiplicative, then f is multiplicative.