Fall 2021 – Math 328K (55385)

Practice midterm exam (with solutions)

Problem 1. Define prime numbers and composite numbers.

Solution 1. A natural number n is composite if it can be written as a product $n = ab$ with $a, b \in \mathbb{N}$ and $a, b > 1$. A natural number n is prime if it is not composite.

Problem 2. What is the last decimal digit of the number 4^{44}?

Solution 2. The question asks for $[4^{44}]$ in $\mathbb{Z}/10\mathbb{Z}$. We get this by the square and multiply algorithm:

$$\begin{aligned}
[4]^2 &= [6] \\
[4]^4 &= ([4]^2)^2 = [6] \\
[4]^5 &= [4] \cdot [4]^4 = [4] \\
[4]^{10} &= ([4]^5)^2 = [6] \\
[4]^{11} &= [4] \cdot [4]^{10} = [4] \\
[4]^{22} &= ([4]^{11})^2 = [6] \\
[4]^{44} &= ([4]^{22})^2 = [6]
\end{aligned}$$

so the last digit of 4^{44} is 6.

Problem 3. The *Fermat numbers* F_n are defined by

$$F_n = 2^{2^n} + 1$$

for all non–negative integers n.

a) Show that $F_{n+1} = F_0 F_1 \cdots F_n + 2$ for all non–negative integers n.

b) Show that F_n and F_m are coprime, for any integers $0 \leq n < m$.

Solution 3.

a) We prove $F_0F_1 \cdots F_n = F_{n+1} - 2$ by induction over n. In the base case $n = 0$, the statement is $2^{2^0} + 1 = 2^{2^1} + 1 - 2$, which is true because both sides are 3. For the inductive step, assume we already know $F_0F_1 \cdots F_n = F_{n+1} - 2$. Then

$$\begin{aligned} F_0F_1 \cdots F_nF_{n+1} &= (F_{n+1} - 2)F_{n+1} = (2^{2^{n+1}} - 1)(2^{2^{n+1}} + 1) \\ &= (2^{2^{n+1}})^2 - 1 = 2^{2^{n+2}} - 1 = F_{n+2} - 2. \end{aligned}$$

This completes the inductive step and the proof.

b) By part a) we can write $F_m = F_0F_1 \cdots F_n \cdots F_{m-1} + 2$, or equivalently

$$F_m - F_0F_1 \cdots F_n \cdots F_{m-1} = 2.$$

Now if $d = (F_m, F_n)$, then d divides the left hand side of this equation, so $d \mid 2$. This means either $d = 1$ or $d = 2$. But the case $d = 2$ cannot occur, since the Fermat numbers are all odd.

Problem 4. Do the following equations have solutions x in $\mathbb{Z}/96\mathbb{Z}$? How many? If there are solutions, list all of them.

a) $9x = 5$

b) $5x = 9$

Solution 4.

a) $(9, 96) = 3$, so this equation has a solution if and only if $3 \mid 5$, which is false. So there are no solutions.

b) $(5, 96) = 1$, so this equation has a solution if and only if $1 \mid 9$, which is true. So there is a unique solution. We can get it either by guessing or by using the extended Euclidean algorithm. In either case we obtain $x = [21]$, since $5 \cdot [21] = [105] = [9]$.

Problem 5. Let $a_1, \dots, a_n, b \in \mathbb{N}$ be positive integers and let $A = a_1a_2 \cdots a_n$. Show that if $(a_i, b) = 1$ for all $i \in \{1, \dots, n\}$, then $(A, b) = 1$.

Solution 5. Assume for contradiction that $(A, b) \neq 1$. Then there exists a prime p such that $p \mid (A, b)$, so $p \mid A$ and $p \mid b$. We proved that if $p \mid a_1 \cdots a_n$ then $p \mid a_i$ for some i. p is therefore a common divisor of a_i and b, which contradicts $(a_i, b) = 1$.