Fall 2021 – Math 328K – 55385

Homework 1 Solutions

Problem 1. Show that, for all sets X, Y and Z,

a) $X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$,

b) $X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$,

c) $X \setminus (X \setminus Y) = X \cap Y$,

d) $X \setminus (Y \cup Z) = (X \setminus Y) \cap (X \setminus Z) = (X \setminus Y) \setminus Z$,

e) $X \setminus (Y \cap Z) = (X \setminus Y) \cup (X \setminus Z)$.

Solution 1. For example by translating these into logical statements and using truth tables.

Problem 2. Using induction, show that for all $n \in \mathbb{N}$ we have

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

Solution 2. First let $n = 1$. Then

$$\sum_{k=1}^{1} k^2 = 1 = \frac{1(1+1)(2 \cdot 1 + 1)}{6}.$$

For the inductive step, let $n \geq 2$ and assume that the equation holds for $n - 1$. That is,

$$\sum_{k=1}^{n-1} k^2 = \frac{(n-1)n(2n-1)}{6}.$$

Then

$$\sum_{k=1}^{n} k^2 = \sum_{k=1}^{n-1} k^2 + n^2 = \frac{(n-1)n(2n-1)}{6} + n^2 = \frac{2n^3 - 3n^2 + n + 6n^2}{6} = \frac{n(n+1)(2n+1)}{6}.$$

This concludes the inductive step and the proof of the equality.

Problem 3. Show that $2^n \geq n^2$ for every integer $n \geq 4$.

Solution 3. We use induction over n starting with 4. In the base case $n = 4$ we have $2^4 = 16 = 4^2$. Now assume that we know $2^n \geq n^2$ and want to show $2^{n+1} \geq (n+1)^2$. We can do this for example by computing

$$(n+1)^2 = n^2 + 2n + 1 \leq n^2 + 2n + n = n^2 + 3n \leq n^2 + n^2 \leq 2 \cdot 2^n = 2^{n+1}.$$

Problem 4. Let R be an ordered ring with $R_+ \subset R$ its set of positive elements. Derive from the axioms that

a) $(-a) \cdot (-b) = ab$ for all $a, b \in R$,

b) $a \cdot a > 0$ for all $a \in R$ unless $a = 0$,

c) if $a, b, c \in R$ with $a > b$ and $b > c$, then $a > c$.

Solution 4.

a) We first show that $(-a) \cdot b = -ab$ for all $a, b \in R$. The inverse $-a$ is defined by the equation $a + (-a) = 0$. Multiplying it by b yields

$$a \cdot b + (-a) \cdot b = (a + (-a)) \cdot b = 0 \cdot b = 0.$$

Here we used the distributive law and that $0 \cdot b = 0$ (we showed this in class). This equation shows that $(-a) \cdot b$ is the additive inverse of ab, that is $(-a) \cdot b = -ab$. A similar proof shows that $a \cdot (-b) = -ab$.

Therefore

$$(-a) \cdot (-b) = -(a \cdot (-b)) = -(-ab) = ab.$$

The last step used that $-(-x) = x$, which follows from $x + (-x) = 0$ and $(-x) + (-(-x)) = 0$.

b) We either have $a = 0$ or $a > 0$ or $a < 0$. If $a > 0$ then $a \cdot a > 0$ by one of the axioms for ordered rings. If $a < 0$, then $-a > 0$, so $(-a) \cdot (-a) > 0$. But $(-a) \cdot (-a) = a \cdot a$ by the previous part, so $a \cdot a > 0$.

c) Assume that $a > b$ and $b > c$. Then $a - b$ and $b - c$ are positive. So

$$a - c = (a - b) + (b - c)$$

is also positive, i.e. $a > c$.

Problem 5. Let R be a ring and $r \in R$. Show that

$$(r-1)\sum_{k=0}^{n} r^k = r^{n+1} - 1.$$

In particular, if R is $\mathbb{Q}$ or $\mathbb{R}$ or $\mathbb{C}$ and $r \neq 1$, then

$$\sum_{k=0}^{n} r^k = \frac{r^{n+1}-1}{r-1}.$$

Solution 5.

$$(r-1)\sum_{k=0}^{n} r^k = \sum_{k=0}^{n} r^{k+1} - \sum_{k=0}^{n} r^k = \sum_{k=1}^{n+1} r^k - \sum_{k=0}^{n} r^k = r^{n+1} - r^0 = r^{n+1} - 1.$$

If R is $\mathbb{Q}$ or $\mathbb{R}$ or $\mathbb{C}$ and $r \neq 1$, we can divide this equation by $r-1$.