Spring 2020 – Math 367K – 53115

Midterm exam (with solutions)

Tuesday, March 10

- Write your name clearly readable on the top of **every page** you write!
- Do not use a red pen.
- No phones, calculators, books, notes, or other aids are permitted.
- When you are done, check once again that every page has your name on it, also write it on this sheet, and use it as a cover.
- Good luck!

Name:

Question 1. Give a definition of a *compact topological space.* Assume we already know basic set theory and what a topological space is, but nothing more than that.

Solution 1. An open cover of a topological space X with topology $\mathcal{T}$ is a set $\mathcal{C} \subset \mathcal{T}$ of open subsets such that $\bigcup \mathcal{C} = X$. The space X is compact if every open cover $\mathcal{C}$ has a finite subcover, meaning a subset $\mathcal{C}' \subset \mathcal{C}$ with $|\mathcal{C}'| < \infty$ and $\bigcup \mathcal{C}' = X$.

Question 2. Let X be a topological space and $A_1, \ldots, A_n \subset X$ be connected subsets such that $\bigcup_{i=1}^n A_i = X$. Assume further that $\bigcap_{i=1}^n A_i \neq \varnothing$. Show that X is connected.

Solution 2. Let $C \subset X$ be a clopen subset. Then $A_i \cap C$ is also clopen in A_i for every i. As the A_i are connected, this means that either $A_i \cap C = A_i$ or $A_i \cap C = \varnothing$.

Choose $x \in \bigcap_{i=1}^n A_i$. If $x \in C$, then $A_i \cap C \neq \varnothing$, so $A_i \cap C = A_i$ for every i. Taking the union over all i, we get that $X \cap C = X$, i.e. $C = X$. On the other hand, if $x \notin C$, then $A_i \cap C \neq A_i$, so $A_i \cap C = \varnothing$ for every i. Again taking the union we get $C = \varnothing$. So the only clopen subsets of X are X and $\varnothing$, i.e. X is connected.

Question 3. Let (X, d) be a metric space. Show that $x \in X$ is an isolated point if and only if

$$\operatorname{dist}(x, X \setminus \{x\}) \neq 0.$$

Remember that $x \in X$ *is called an isolated point if* $\{x\}$ *is open in* X*. For* $x \in X$ *and* $A \subset X$ *the distance* $\operatorname{dist}(x, A)$ *is defined by* $\operatorname{dist}(x, A) = \inf_{y \in A} d(x, y)$.

Solution 3. Suppose that x is isolated, i.e. $\{x\}$ is open. Then there is some $\varepsilon > 0$ with $B_\varepsilon(x) \subset \{x\}$. In other words, $d(x, y) \geq \varepsilon$ for every $y \in X \setminus \{x\}$. So $\operatorname{dist}(x, X \setminus \{x\}) \geq \varepsilon > 0$.

Conversely, suppose $\operatorname{dist}(x, X \setminus \{x\}) = d \neq 0$. Since $d = \inf_{y \in X \setminus \{x\}} d(x, y)$, we must have $d(x, y) \geq d$ for all $y \in X \setminus \{x\}$. So $B_d(x) = \{x\}$. Since these balls are open, x is isolated.

Note that technically we should also consider the case that X has only one element. In this case, x would be isolated, and $\operatorname{dist}(x, X \setminus \{x\}) = \infty$ (or undefined).

Question 4. Let $\mathcal{B} \subset \mathcal{P}(\mathbb{R}^2)$ be defined by

$$\mathcal{B} = \{Q_r(x) \mid x \in \mathbb{R}^2, r \in \mathbb{R}\}, \qquad Q_r(x) = (x_1 - r, x_1 + r) \times (x_2 - r, x_2 + r)$$

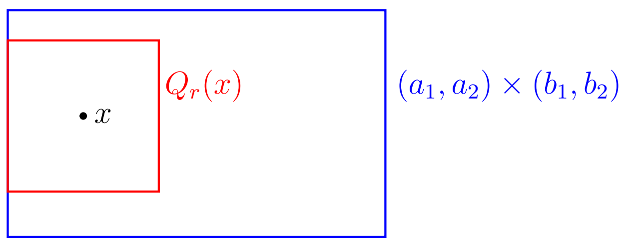
Show that $\mathcal{B}$ is a basis of the standard topology on $\mathbb{R}^2$. You can assume we already know that it is a basis of *some* topology on $\mathbb{R}^2$.

Solution 4. Since we already know that $\mathcal{B}$ is a basis, we only have to show that the topology it generates is the standard topology. So let $\mathcal{T}$ denote the topology generated by $\mathcal{B}$ and $\mathcal{T}_s$ the standard topology. It is generated for example by the set of open boxes, which we call $\mathcal{B}_s$.

Clearly $\mathcal{B} \subset \mathcal{B}_s$ and therefore $\mathcal{T} \subset \mathcal{T}_s$. On the other hand, take some open box $(a_1, a_2) \times (b_1, b_2) \in \mathcal{B}_s$ and assume $x \in (a_1, a_2) \times (b_1, b_2)$, i.e. $a_1 < x_1 < b_1$ and $a_2 < x_2 < b_2$. Set

$$r = \min\{x_1 - a_1, x_2 - a_2, b_1 - x_1, b_2 - x_2\} > 0.$$

Then $Q_r(x) \subset (a_1, a_2) \times (b_1, b_2)$. Since we find such a neighborhood for every $x \in (a_1, a_2) \times (b_1, b_2)$, we can write this set as a union of elements of $\mathcal{B}$. So $\mathcal{B}_s \subset \mathcal{T}$ and therefore $\mathcal{T}_s \subset \mathcal{T}$.



Question 5. Let X be a compact topological space and let $f\colon X \to \mathbb{R}^+$ and $g\colon X \to \mathbb{R}^+$ be continuous positive real–valued functions.

a) Show that the set $A = \{x \in X \mid f(x) = g(x)\}$ is compact (in the subspace topology).

b) Show that there is a constant $C > 1$ such that

$$\frac{1}{C} f(x) \leq g(x) \leq C f(x) \qquad \forall x \in X.$$

Solution 5. We consider the function $h\colon X \to \mathbb{R}^+$ defined by $h(x) = g(x)/f(x)$. This function is continuous because the map $(\mathbb{R}^+)^2 \to \mathbb{R}^+, (u, v) \mapsto u/v$ is continuous (which is known from analysis).

Now the set A from a) equals $h^{-1}(\{1\})$, so it is closed. As a closed subset of the compact space X, A is compact in the subspace topology.

The image $h(X)$ is a compact subset of $\mathbb{R}^+$, so it has a minimum $m > 0$ and a maximum $M > 0$. Choose $C = \max\{1/m, M\} > 0$. Then $1/C \leq m$ and $M \leq C$, so $h(X) \subset [m, M] \subset [1/C, C]$ and therefore

$$\frac{1}{C} \leq \frac{g(x)}{f(x)} \leq C \qquad \forall x \in X.$$