Spring 2020 – Math 367K – 53115

Final exam – solutions

Thursday, May 14 to Saturday, May 16

Before looking at the questions, carefully read these instructions:

- Before you start, go to a place where you can work undisturbed for three hours, and take a pen or pencil and a stack of empty paper with you.
- You can print this exam out or view it on a computer/tablet/phone/etc., but do not use that device for anything else while solving the exam (if it's a phone, set it to airplane mode)!
- Also, no calculators, books, notes, the internet or any other aids are allowed.
- You have **three hours** time to work on the exam. **Set a timer!** The time starts when you first look at the questions (on the next pages).
- **I trust you not to cheat.** Obviously, there would be plenty of possibilities in this format. The only thing that really prevents you from doing that is your honesty.
- After you're done, or the time is over, scan your solutions and submit them in Canvas (just like the homework). Please submit them by Saturday night.
- Good luck!

Question 1. Which of the following statements are true? You do **not** need to give an argument for your answers.

☐ Every connected space is path connected.

false: the topologist's sine curve is a counterexample.

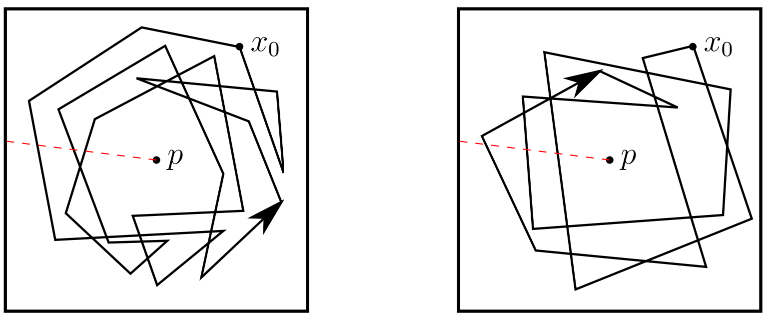
☐ Every path connected space is connected.

true: we proved it in class.

☐ Every connected space is locally connected.

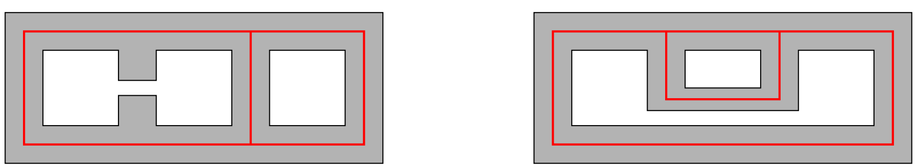
false: the topologist's sine curve is also a counterexample for this.

☐ These two paths at x_0 are homotopic in $\mathbb{R}^2 \setminus \{p\}$:



true: the rotation numbers of the two paths agree, which can be easily checked by drawing a radial line (shown in red) and taking the signed count of its crossings with the path: in both pictures, the path crosses the red line two times in downward direction, and once in upward direction, so the rotation number is 1. the rotation number is an isomorphism from the fundamental group to $\mathbb{Z}$, so if it is the same, the paths are homotopic.

☐ The two gray regions in $\mathbb{R}^2$ have isomorphic fundamental groups:



true: the gray regions are deformation retracts of the red graphs, which are (almost) wedges of two circles, so the fundamental group in both cases is the free group in two generators.

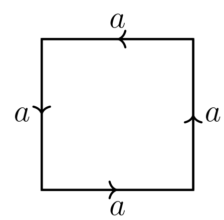
☐ Every second countable T_4 space is metrizable.

true: this is the statement of the Urysohn metrization theorem.

☐ Every metrizable topology is induced by a unique metric.

false: for example, $d(x, y) = \min\{|x - y|, 1\}$ defines a metric on $\mathbb{R}$ which induces the same topology as the standard metric.

☐ The fundamental group of a square glued in this way is isomorphic to $\mathbb{Z}$:



false: the fundamental group of this space (called a *dunce cap*) is $\langle a \mid a^4 = 1 \rangle \cong \mathbb{Z}/4\mathbb{Z}$.

☐ Let $X = S^2 \setminus \{x, y, z\}$ with x, y, z being pairwise distinct. Then the fundamental group of X is the free group in 3 generators.

false: the sphere with three points removed is homeomorphic to $\mathbb{R}^2$ with two points removed, so its fundamental group is the free group in *two* generators.

Question 2. Let X be a metric space with metric d and $K \subset U \subset X$ with K compact and U open. Show that there exists $\varepsilon > 0$ with $N_\varepsilon(K) \subset U$, where

$$N_\varepsilon(K) = \{x \in X \mid \operatorname{dist}(x, K) < \varepsilon\}.$$

Solution 2. *This question was identical to Exercise 2 on Homework Sheet 8.* Let $x \in K$. Since U is open, there is an $\varepsilon_x > 0$ such that $B_{\varepsilon_x}(x) \subset U$. The collection of balls half that radius, i.e. $\{B_{\varepsilon_x/2}(x) \mid x \in K\}$ is an open cover of K, so it has a finite subcover. This means there are $x_1, \dots, x_n$ and $\varepsilon_1, \dots, \varepsilon_n$ such that

$$K \subset B_{\varepsilon_1/2}(x_1) \cup \dots \cup B_{\varepsilon_n/2}(x_n) \subset U.$$

Let $\varepsilon = \min\{\varepsilon_1/2, \dots, \varepsilon_n/2\}$. Then for every $y \in N_\varepsilon(K)$ we find $z \in K$ with $d(y, z) = \operatorname{dist}(y, K) < \varepsilon$ (1c from homework 6, though something weaker than that would suffice). Since $z \in B_{\varepsilon_i/2}(x_i)$ for some i, $d(y, x_i) \le d(y, z) + d(z, x_i) < \varepsilon + \varepsilon_i/2 \le \varepsilon_i$, hence $y \in B_{\varepsilon_i}(x_i) \subset U$. So we get $N_\varepsilon(K) \subset U$.

Question 3. Let X and Y be metric spaces and $f\colon X \to Y$ a continuous map with the property that $f^{-1}(K)$ is compact for every compact subset $K \subset Y$. Show that

a) If $(x_n)_{n\in\mathbb{N}}$ is a sequence in X and $f(x_n)$ converges, then a subsequence of x_n converges.

   Hint: consider the set $K = \{f(x_n) \mid n \in \mathbb{N}\} \cup \{\lim_n f(x_n)\}$.

b) $f(C)$ is closed for every closed set $C \subset X$.

Solution 3.

a) $K = \{f(x_n) \mid n \in \mathbb{N}\} \cup \{\lim_n f(x_n)\}$ is a compact set (every sequence in K either has a constant subsequence or one converging to $\lim_n f(x_n)$), so $f^{-1}(K)$ is compact. x_n is a sequence in $f^{-1}(K)$, so it has a convergent subsequence.

b) Let $C \subset X$ be closed and x_n a sequence in C such that $f(x_n)$ converges to some limit $y \in Y$. By a), a subsequence of x_{n_k} of x_n converges, say to x, and $x \in C$ because C is closed. Then $f(x_{n_k}) \to f(x)$ since f is continuous, but also $f(x_n) \to y$ by assumption. Limits are unique in a metric space, so $y = f(x) \in f(C)$.

   We showed that the limit of an arbitrary convergent sequence in $f(C)$ is again in $f(C)$, i.e. that $f(C)$ is closed.

Question 4. Let X be a topological space and $A \subset X$. Assume that there exists a continuous map $r\colon X \to A$ with the property that $r(a) = a$ for all $a \in A$.

Show that, if $x_0 \in A$, the group homomorphism

$$\iota_*\colon \pi_1(A, x_0) \to \pi_1(X, x_0)$$

induced by the inclusion map $\iota\colon A \to X$ is injective.

Solution 4. The map r induces a map

$$r_*\colon \pi_1(X, x_0) \to \pi_1(A, x_0)$$

on fundamental groups, and

$$r_* \circ \iota_* = (r \circ \iota)_* = (\mathrm{id}_A)_* = \mathrm{id}_{\pi(A,x_0)}\,.$$

So ι_* has a left inverse and is therefore injective.

Question 5. Consider the space $X = (D \times \{0, 1\})/\!\sim$ where

$$D = \{(x_1, x_2) \mid x_1^2 + x_2^2 \leq 1\} \subset \mathbb{R}^2$$

is the disk the equivalence relation $\sim$ is defined by $(x, 0) \sim (x, 1)$ for all $x \in S^1 = \partial D$. Show that X is homeomorphic to S^2 by explicitly constructing a homeomorphism.

Solution 5. The map

$$\widetilde{f}\colon D \times \{0, 1\} \to S^2, \qquad (x, y, i) \mapsto (x, y, (-1)^i \sqrt{1 - x^2 - y^2})$$

is continuous, surjective, actually maps into S^2, and if $(x, y) \in S^1$, then $\widetilde{f}(x, y, 0) = (x, y, 0) = \widetilde{f}(x, y, 1)$. So it determines a continuous and surjective function $f\colon X \to S^2$. It is also injective: if $\widetilde{f}(x_1, y_1, i_1) = \widetilde{f}(x_2, y_2, i_2)$, then $x_1 = x_2$, $y_1 = y_2$ and

$$(-1)^{i_1}\sqrt{1 - x_1^2 - y_1^2} = (-1)^{i_2}\sqrt{1 - x_2^2 - y_2^2}.$$

This implies $i_1 = i_2$, so $(x_1, y_1, i_1) = (x_2, y_2, i_2)$, or $1 - x_1^2 - y_1^2 = 0$, so $(x_1, y_1) = (x_2, y_2) \in S^1$. In either case $(x_1, y_1, i_1) \sim (x_2, y_2, i_2)$, so f is injective.

Finally, $D \times \{0, 1\}$ is compact, so X is compact, and as a bijective continuous map from a compact space to a Hausdorff space, f is a homeomorphism.