Exercise Sheet 1

due Thursday, January 30, 9:30

Exercise 1. Let X, Y be sets, $A, A' \subset X$ and $B, B' \subset Y$, and $f\colon X \to Y$ a map. Show that

a) $A \subset f^{-1}(f(A))$ and equality holds if f is injective,

b) $f(f^{-1}(B)) \subset B$ and equality holds if f is surjective,

c) $f^{-1}(B \cup B') = f^{-1}(B) \cup f^{-1}(B')$,

d) $f^{-1}(B \cap B') = f^{-1}(B) \cap f^{-1}(B')$,

e) $f(A \cup A') = f(A) \cup f(A')$,

f) $f(A \cap A') \subset f(A) \cap f(A')$.

Find an example where $f(A \cap A') \neq f(A) \cap f(A')$.

Exercise 2. Prove that the composition of injective/surjective/bijective maps is injective/surjective/bijective.

Exercise 3 (Cantor–Bernstein–Schröder)**.** Let $f\colon A \to B$ and $g\colon B \to A$ be injective maps. We define recursively the sets

$$C_0 = A \setminus g(B), \qquad C_{n+1} = g(f(C_n)), \qquad C = \bigcup_{n=0}^{\infty} C_n$$

and a new map $h\colon A \to B$ by

$$h(x) = \begin{cases} f(x) & \text{if } x \in C, \\ g^{-1}(x) & \text{if } x \notin C, \end{cases}$$

where the preimage $g^{-1}(x)$ is well–defined since g is injective and $x \in g(B)$ in that case (check that!). Show that h is bijective.

Conclude that if $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.

Exercise 4. To solve this, you might need some tools from Real Analysis.

a) Show that the map

$$f\colon \mathbb{R} \to \mathcal{P}(\mathbb{Q}), \quad x \mapsto \{q \in \mathbb{Q} \mid q \leq x\}$$

is injective.

b) Let $\{0,1\}^{\mathbb{N}}$ denote the set of infinite sequences $(a_1, a_2, \dots)$ with $a_i \in \{0,1\}$ for all i. Show that the map

$$g\colon \{0,1\}^{\mathbb{N}} \to \mathbb{R}, \quad (a_i)_{i\in\mathbb{N}} \mapsto 2\sum_{i=1}^{\infty} 3^{-i} a_i$$

is injective. Its image $C = g(\{0,1\}^{\mathbb{N}})$ is called the *Cantor set*. We will meet it again later in the course.

c) Show that $|\{0,1\}^{\mathbb{N}}| = |\mathcal{P}(\mathbb{N})| = |\mathcal{P}(\mathbb{Q})|$.

d) Use the previous parts to show

$$|C| = |\mathbb{R}| = |\mathcal{P}(\mathbb{N})|.$$