Spring 2020 – Math 367K – 53115

Exercise Sheet 2

due Thursday, February 6, 9:30

Exercise 1. Let X be a topological space with basis $\mathcal{B}$ and $Y \subset X$. Show that

$$\mathcal{B}_Y = \{B \cap Y \mid B \in \mathcal{B}\}$$

is a basis for the subspace topology on Y.

Exercise 2. Let

$$\mathcal{B}_B = \{B_\varepsilon(x) \mid \varepsilon \in \mathbb{R}^+, x \in \mathbb{R}^n\}, \qquad B_\varepsilon(x) = \{y \in \mathbb{R}^n \mid \sum_{i=1}^n (y_i - x_i)^2 < \varepsilon\}$$

be the set of open balls in $\mathbb{R}^n$ and let

$$\mathcal{B}_R = \{R(u,v) \mid u, v \in \mathbb{R}^n\}, \qquad R(u,v) = \{x \in \mathbb{R}^n \mid \forall i \leq n\colon u_i < x_i < v_i\}$$

be the set of open boxes (or rectangles) in $\mathbb{R}^n$.

a) Show that $\mathcal{B}_B$ and $\mathcal{B}_R$ are each a basis of a topology on $\mathbb{R}^n$.

b) Show that the topologies generated by $\mathcal{B}_B$ and $\mathcal{B}_R$ are equal.

Exercise 3. Show that the countable set

$$\{(a,b) \times (c,d) \mid a,b,c,d \in \mathbb{Q}, a < b, c < d\}$$

is a basis for the standard topology on $\mathbb{R}^2$.

Exercise 4. Consider the following topologies on $\mathbb{R}$:

$\mathcal{T}_1$ = the standard topology
$\mathcal{T}_2$ = the finite complement topology
$\mathcal{T}_3$ = the lower limit topology, having all sets $[a,b)$ as basis
$\mathcal{T}_4$ = the upper limit topology, having all sets $(a,b]$ as basis
$\mathcal{T}_5$ = the topology having all sets $(-\infty, a)$ as basis

Determine which are these topologies are finer / coarser than which of the others.

Definition of the *finite complement topology*: the open sets are all sets whose complement in X is finite (and the empty set).