Spring 2020 – Math 367K – 53115

Exercise Sheet 4

due Thursday, February 20, 9:30

Exercise 1. Let X, Y be topological spaces, $f, g\colon X \to Y$ functions from X to Y, $U \subset X$ an open set, and $x \in U$. Assume that $f|_U = g|_U$. Show that f is continuous at x if and only if g is continuous at x.

This shows that whether a function is continuous at x only depends on its behavior in an (arbitrarily small) open neighborhood of x. We say that continuity at a point is a *local property*.

Exercise 2. Let X, Y be topological spaces and $A \subset X$.

a) Show that the subspace topology on A is the coarsest topology such that the inclusion map $\iota\colon A \to X$ is continuous.

b) Show that the product topology on $X \times Y$ is the coarsest topology such that the projections $\pi_1\colon X \times Y \to X$ and $\pi_2\colon X \times Y \to Y$ are continuous.

Exercise 3. Let $f\colon X \to Y$ be a continuous map between topological spaces X and Y. Show that if a sequence $x_n \in X$ converges to x, then $f(x_n)$ converges to $f(x)$.

Exercise 4. Let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of connected subspaces of X, with the property that $A_n \cap A_{n+1} \neq \varnothing$ for every $n \in \mathbb{N}$. Show that $\bigcup_{n\in\mathbb{N}} A_n$ is connected.

Exercise 5. A space X is called *totally disconnected* if the only connected subspaces are the one–point sets.

a) If X has the discrete topology, show that it is totally disconnected.

b) Show that $\mathbb{Q} \subset \mathbb{R}$ is totally disconnected.

c) Show that the Cantor set $C \subset \mathbb{R}$ is totally disconnected (see exercise sheet 1 for the definition).

Hint for c): To get an intuition for the Cantor set, convince yourself that it can be constructed iteratively in the following way: take the unit interval $[0, 1]$ *and remove the (open) middle third. The remainder is the set* $[0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$*. Now perform the same operation on each of its parts and obtain* $[0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$*. If we continue this procedure infinitely and then take the intersection over all steps, we get* C*. Here is a picture of the first 7 steps:*