Spring 2020 – Math 367K – 53115

Exercise Sheet 5

due Tuesday, March 3, 9:30

Exercise 1. Let X be a topological space and Y a compact Hausdorff space. Show that a function $f\colon X \to Y$ is continuous if and only if its graph

$$\mathrm{gr}(f) = \{(x, f(x)) \mid x \in X\}$$

is a closed subset of $X \times Y$.

Hint: Show that the projection $X \times Y \to X$ maps closed sets to closed sets.

Exercise 2. Consider $\mathbb{R}$ with the finite complement topology. Show that every subset is compact.

Exercise 3. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two topologies on a set X which are both compact and Hausdorff. Show that either $\mathcal{T}_1 = \mathcal{T}_2$ or the two topologies are incomparable.

Exercise 4. Let X be compact, Y a Hausdorff space and $f\colon X \to Y$ continuous.

a) Show that f is a *closed map*, i.e. the image of every closed set is closed.

b) Show that if f is bijective, it is a homeomorphism.

Exercise 5. Let X be a compact space and let $(C_i)_{i\in\mathbb{N}}$ be a sequence of non–empty closed subsets such that $C_{i+1} \subset C_i$ for every i. Show that the intersection $\bigcap_{i\in\mathbb{N}} C_i$ is non–empty.

Hint: Assume the contrary and look at the complements.

Definition 1. A *compactification* of a topological space X is a pair (Y, f) consisting of a topological space Y and a continuous map $f\colon X \to Y$ such that

- Y is compact,
- f is a homeomorphism onto $f(X)$,

- and $f(X)$ is dense in Y, i.e. $\overline{f(X)} = Y$.

Definition 2. A space X is *locally compact* if every $x \in X$ has an open neighborhood U and a compact set $K \subset X$ such that $U \subset K$.

Exercise 6.

a) Let $X = \mathbb{R}$ with the standard topology and let $Y = \mathbb{R} \cup \{-\infty, \infty\}$ equipped with the topology generated by all open intervals (a, b) as well as the sets $[-\infty, a)$ and $(a, \infty]$ for any $a \in \mathbb{R}$. Let $f\colon \mathbb{R} \to Y$ be the identity map. Show that (Y, f) is a compactification of $\mathbb{R}$.

b) Let X be a locally compact, but non–compact Hausdorff space. Let $Y = X \cup \{\infty\}$, and equip Y with a topology as follows: a subset $A \subset Y$ is open if

   - either $\infty \notin A$ and A is open as a subset of X,
   - or $\infty \in A$ and $Y \setminus A$ is a compact subset of X.

   Let $f\colon X \to Y$ be the identity map. Show that (Y, f) is a compactification of X. It is called the *one–point compactification* of X.

c) Show that the one–point compactification of $\mathbb{R}$ is homeomorphic to S^1.