Exercise Sheet 6

due Thursday, March 12, 9:30

Exercise 1. Let (X, d) be a metric space. For a subset $A \subset X$ and $x \in X$ define

$$\operatorname{dist}(x, A) = \inf_{y \in A} d(x, y).$$

a) Show that the map $X \to [0, \infty), x \mapsto \operatorname{dist}(x, A)$ is continuous.

b) Show that $\operatorname{dist}(x, A) = 0$ if and only if $x \in \overline{A}$.

c) Assume A is compact and let $x \in X$. Show that there exists $y \in A$ with

$$d(x, y) = \operatorname{dist}(x, A).$$

Give a counterexample to this when A is not compact.

Exercise 2. Show that every subspace and every (finite) product of first/second countable spaces are first/secound countable.

Exercise 3. Let (X, d) be a metric space. Show that

$$\overline{d}(x, y) = \begin{cases} d(x, y) & \text{if } d(x, y) < 1 \\ 1 & \text{otherwise} \end{cases}$$

defines another metric $\overline{d}$ on X, and that d and $\overline{d}$ induce the same topology.

Exercise 4. A topological space X is called *separable* if it contains a countable dense subset. Show that a metric space X is second countable if and only if it is separable.

Exercise 5. An *isolated point* in a topological space X is a point $x \in X$ so that $\{x\}$ is open.

a) Show that X has the discrete topology if and only if every point of X is isolated.

b) Show that if X is compact and every point in X is isolated, then X is finite.

c) Show that a subspace of $\mathbb{R}^n$ can only have countably many isolated points.

Exercise 6. Recall the definition of the Cantor set C from Sheets 1 and 4. Show that

a) C is compact,

b) C is metrizable,

c) C has no *isolated points*, that is no one–point subset $\{x\} \subset C$ is open,

d) Equip the set $\{0,1\}^{\mathbb{N}}$ with the *infinite product topology*, the coarsest topology which makes all projections

$$\pi_i \colon \{0,1\}^{\mathbb{N}} \to \{0,1\}, \quad (x_n)_{n\in\mathbb{N}} \mapsto x_i$$

continuous. Here $\{0,1\}$ carries the discrete topology. Show that this makes the map $\{0,1\}^{\mathbb{N}} \to C$ from Exercise Sheet 1 a homeomorphism.