Exercise Sheet 7

due Thursday, April 2

Exercise 1. Let X be a topological space and (x_n) a sequence in X.

a) Show that if there is an $x \in X$ such that every subsequence of (x_n) has a subsequence converging to x, then (x_n) converges to x.

b) Find a counterexample to the following statement: if every subsequence of (x_n) has a converging subsequence, then (x_n) converges.

Exercise 2. Let d and d' be two metrics on a set X, and $C \geq 1$ a constant such that

$$\frac{1}{C}\, d(x,y) \leq d'(x,y) \leq C\, d(x,y)$$

for all $x, y \in X$. Show that both metrics induce the same topology on X.

Exercise 3. Let I be some (possibly infinite) index set, and X_i a topological space for every $i \in I$. We consider the Cartesian product

$$X = \prod_{i \in I} X_i.$$

The *product topology* on $\prod_{i \in I} X_i$ is defined to be the coarsest topology which makes all the projection maps $\pi_i \colon X \to X_i$ for $i \in I$ continuous. As a convention, we write x_i for $\pi_i(x)$.

a) Show that the sets of the form $\prod_{i \in I} U_i$ with $U_i \subset X_i$ open do *not* in general form a basis of a product topology, but generate a finer topology. What would be a basis of the product topology?

b) Assume that every X_i is metrizable with a metric d_i. For every $\epsilon > 0$, $x \in X$, and every finite subset $J \subset I$ let

$$B_\varepsilon^J(x) = \{y \in X \mid d_i(x_i, y_i) < \varepsilon \ \forall i \in J\}.$$

Show that the set of all such $B_\varepsilon^J(x)$ (for all ε, J, and x) are a basis of the product topology on X.

c) Now assume in addition that $I = \mathbb{N}$. Define

$$d(x,y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{d_i(x_i, y_i)}{1 + d_i(x_i, y_i)},$$

with $x, y \in X$. Show that d is a metric.

d) Show that for every $\varepsilon > 0$ and $J \subset I$ finite there exists $\delta > 0$ with $B_\delta(x) \subset B^J_\varepsilon(x)$. (Here $B_\delta(x)$ denotes the δ–ball in the metric d.)

e) Show that for every $\varepsilon > 0$ there exists a finite $J \subset I$ and $\delta > 0$ such that $B^J_\delta(x) \subset B_\varepsilon(x)$.

f) Deduce that the metric topology on X induced by d is equal to the product topology.

g) As an example for a product of uncountably many factors consider the set

$$X = \mathbb{R}^\mathbb{R} = \{f \mid f \text{ is a function } f\colon \mathbb{R} \to \mathbb{R}\}.$$

We equip it with the product topology, i.e. the coarsest topology such that the projection maps

$$\pi_x\colon \mathbb{R}^\mathbb{R} \to \mathbb{R}, \quad f \mapsto f(x)$$

are continuous for every $x \in \mathbb{R}$. Show that $\mathbb{R}^\mathbb{R}$ is not first countable and hence not metrizable.