Spring 2020 – Math 367K – 53115

Exercise Sheet 8

due Thursday, April 9

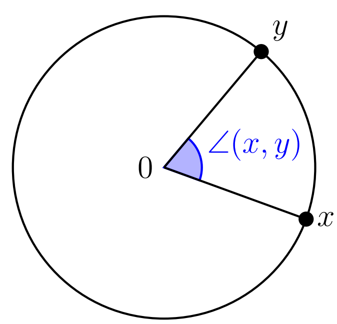
Exercise 1. Consider the circle

$$S^1 = \{x \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = 1\}.$$

For two points $x, y \in S^1$, let d be their Euclidean distance

$$d(x,y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$$

and $\angle(x,y)$ the angle between the two points as seen from the origin (as a number in $[0,\pi]$).



Show that d and $\angle$ both define metrics on S^1 and both these metrics induce the subspace topology on S^1 (as a subset of $\mathbb{R}^2$).

Exercise 2. Let X be a metric space with metric d and $K \subset U \subset X$ with K compact and U open. Show that there exists $\varepsilon > 0$ with $N_\varepsilon(K) \subset U$, where

$$N_\varepsilon(K) = \{x \in X \mid \operatorname{dist}(x,K) < \varepsilon\}.$$

Exercise 3. Let X be a compact space and Y a metric space with metric d. Let $C(X,Y)$ be the set of all continuous functions from X to Y.

a) Show that

$$d_C(f,g) = \sup_{x\in X} d(f(x),g(x))$$

defines a metric d_C on $C(X,Y)$.

b) For a compact subset $K \subset X$ and an open subset $U \subset Y$ define

$$V_{K,U} = \{f \in C(X,Y) \mid f(K) \subset U\}.$$

Show that the sets $V_{K,U}$ are open in the metric d_C.

c) Show that

$$\mathcal{S} = \{V_{K,U} \mid K \subset X \text{ compact and } U \subset Y \text{ open}\}$$

is a subbasis of the metric topology on $C(X,Y)$ with the metric d_C.

Hint: You will need to show that $f \in V_{K_1,U_1} \cap \cdots \cap V_{K_n,U_n} \subset B_\varepsilon(f)$ for some suitable choices of $K_1,\dots,K_n$ and $U_1,\dots,U_n$. Choose the K_i so that they cover X but every one of them is small. Then choose the U_i a little bit bigger than $f(K_i)$.

d) Conclude that the topology on $C(X,Y)$ does not depend on the metric on Y, but only the topology on Y. It is called the *compact open topology*, and can even be defined using $\mathcal{S}$ without assuming that Y is metrizable.

Exercise 4. Let (X,d_X) and (Y,d_Y) be metric spaces. A map $f\colon X \to Y$ is called *uniformly continuous* if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $d_X(x_1,x_2) < \delta$ implies $d_Y(f(x_1),f(x_2)) < \varepsilon$ for all $x_1,x_2 \in X$.

a) Show that every uniformly continuous map is continuous.

b) Show that the converse holds if X is compact, i.e. every continuous map is uniformly continuous.

Hint: Given ε the continuity gives you a δ for every $x \in X$. We just have to show that a single δ works for every x. It might be helpful to use the following fact which we proved as part of Theorem 3.6: if $\mathcal{C}$ is an open cover of a compact space X, then there exists an $r > 0$ such that every ball of radius r is contained in a single set $U \in \mathcal{C}$ of the cover.

c) Give an example of a map which is continuous but not uniformly continuous.