Spring 2020 – Math 367K – 53115

Exercise Sheet 9

due Thursday, April 16

Definition 1. A map is called *open* if the image of every open set is open, and *closed* if the image of every closed set is closed.

Definition 2. A group action $G \times X \to X$ on a topological space X is *continuous* if for every $g \in G$ the map $X \to X, x \mapsto gx$ is continuous.

Exercise 1. Let X, Y be topological spaces and $f\colon X \to Y$ a continuous surjective map. Show that

a) If f is a quotient map, then $Y \cong X/\sim$ where $\sim$ is the equivalence relation defined by $x \sim x' \Leftrightarrow f(x) = f(y)$.

b) If f is open, then f is a quotient map.

c) If f is closed, then f is a quotient map.

d) Find an example of a quotient map which is neither open nor closed.

Exercise 2. Let $G \times X \to X$ be a continuous group action. Show that the projection $p\colon X \to X/G$ is open.

Exercise 3. Let $G \times X \to X$ be a continuous group action on a topological space, $\sim$ the associated equivalence relation, and $p\colon X \to X/G$ the natural projection. Further let $A \subset X$ be a subset with the property that $p(A) = p(X)$.

a) Show that $A/\sim \cong X/G$ (the symbol $\cong$ means *is homeomorphic to*).

b) Show that

$$I/\approx \cong \mathbb{R}/\mathbb{Z},$$

where $\approx$ is the equivalence relation defined by $0 \approx 1$ and $\mathbb{Z}$ acts on $\mathbb{R}$ by addition.

c) Show that
$$I^2/{\simeq} \cong \mathbb{R}^2/\mathbb{Z}^2,$$
where $\simeq$ is defined by $(0,x) \simeq (1,x)$ and $(x,0) \simeq (x,1)$ for all $x \in I$, and the group $(\mathbb{Z}^2,+)$ acts on $\mathbb{R}^2$ by (vector) addition.

Exercise 4. Let $R > r > 0$. Show that the map

$$f\colon \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{R}^3, \quad [(s,t)] \mapsto \begin{pmatrix} R\cos(2\pi s) + r\sin(2\pi t)\cos(2\pi s) \\ R\sin(2\pi s) + r\sin(2\pi t)\sin(2\pi s) \\ r\cos(2\pi t) \end{pmatrix}$$

is well–defined and an imbedding and that its image is the set $(x,y,z) \in \mathbb{R}^3$ solving the equation
$$\left(\sqrt{x^2+y^2} - R\right)^2 + z^2 = r^2.$$
Convince yourself that this is indeed the torus.