

## Exercise Sheet 10

*due Thursday, April 23*

**Exercise 1** (Proof writing exercise, due April 28). To practice your proof writing skills, pick one of the longer exercises from the past homework and write up your solution again, as clearly and detailed as possible. Even if your solution got all points, that only means it's correct. But it is always possible to improve the exposition!

- Pick your favorite one of the following exercises (sheet.exercise): 7.3, 8.3 or 9.4.
- Fix any mathematical mistakes, if they existed in your original solution.
- Critically review your write-up, add missing details and improve sections which are not completely clear.
- Actually, it can be better to start over and write everything another time. This makes sure that you have really revisited every sentence. Maybe you find a simpler argument for some part, then use that instead.
- The goal is to end up with a proof which would be understandable for *everyone* else in the class, even if they didn't try to solve the problem themselves.
- If you think your solution meets this requirement, try to improve it even further. If a section is very technical and hard to read, maybe explain the rough idea first. Maybe add pictures. Do whatever you think makes it easier to understand.
- When you're done, find a partner and let them read your write-up. Incorporate their suggestions (if you agree with them). Indicate who read your solution on the submission.
- Finally, submit your proof on Canvas until April 28.
- This counts outside of the usual homework, and will make up a higher percentage of your final grade. Other than the usual homeworks, this is only about form, and while your argument should still be correct, that's not the main focus.

We also have some “normal” exercises this week, but it’s slightly shorter than usual.

**Definition 1.** Let  $f, g: X \rightarrow Y$  be homeomorphisms of topological spaces  $X$  and  $Y$ . They are said to be *isotopic* if there is a continuous family  $f_t: X \rightarrow Y$  of maps for  $t \in I$  such that  $f = f_0$  and  $g = f_1$ , and  $f_t$  is a homeomorphism for every  $t \in I$ . Here “continuous family” means that the map

$$I \times X \rightarrow Y, \quad (t, x) \mapsto f_t(x)$$

is continuous.

**Exercise 2.** Show that every homeomorphism  $f: [0, 1] \rightarrow [0, 1]$  is isotopic to either the identity or the map  $t \mapsto 1 - t$ .

**Exercise 3.** For a homeomorphism  $f: I \rightarrow I$  we consider the space  $I^2/\sim_f$  where the equivalence relation  $\sim_f$  is given by  $(0, x) \sim_f (1, f(x))$  for all  $x \in I$ . In other words, the square with the two vertical edges glued together via  $f$ .

Show that if  $f$  and  $g$  are isotopic, then  $I^2/\sim_f$  is homeomorphic to  $I^2/\sim_g$ .

**Exercise 4.** Recall that the *real projective plane*  $\mathbb{RP}^2$  is the quotient of the action of the group  $(\mathbb{R} \setminus \{0\}, +)$  on  $\mathbb{R}^3 \setminus \{0\}$  via scalar multiplication. In other words, if  $x, y \in \mathbb{R}^3 \setminus \{0\}$ , then  $[x] = [y] \in \mathbb{RP}^2$  if and only if  $x$  and  $y$  lie on the same line through the origin in  $\mathbb{R}^3$ . We write

$$p: \mathbb{R}^3 \setminus \{0\} \rightarrow \mathbb{RP}^2, \quad x \mapsto [x]$$

for the natural projection.

In this exercise we want to show that the space obtained by polygon gluing with the labelling scheme *abab* is the real projective plane  $\mathbb{RP}^2$ .

a) Show that the map

$$f: [-1, 1]^2 \rightarrow D, \quad x \mapsto \frac{\max\{|x_1|, |x_2|\}}{\sqrt{x_1^2 + x_2^2}} x$$

is a homeomorphism, where  $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$  is the unit disk.

b) Let

$$H = \{(x, y, z) \in S^2, z \geq 0\}$$

be the upper half-sphere. Show that there is a homeomorphism  $g: D \rightarrow H$  with

$$g(x, y) = (x, y, 0) \quad \forall (x, y) \in S^1 \subset D.$$

c) Consider the composition

$$F: [-1, 1]^2 \xrightarrow{f} D \xrightarrow{g} H \xrightarrow{\iota} \mathbb{R}^3 \setminus \{0\} \xrightarrow{p} \mathbb{RP}^2.$$

where  $\iota: H \rightarrow \mathbb{R}^3 \setminus \{0\}$  is just the inclusion map. Show that  $F$  a quotient map, and therefore  $\mathbb{RP}^2 \cong [-1, 1]^2 / \sim$  for the equivalence relation defined by  $x \sim y \Leftrightarrow F(x) = F(y)$ . Show that this equivalence relation can be described as gluing along the labelling scheme  $abab$ .

*Hint for all parts: you can save some work by looking at Exercise 4 on Sheet 5. Also, these pictures might be helpful:*

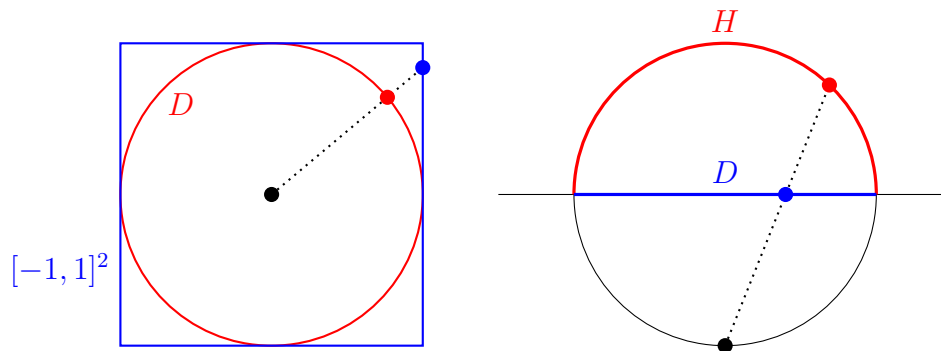


Figure 1: Pictures which might help you understand the homeomorphisms in a and b.