Spring 2020 – Math 367K – 53115

Exercise Sheet 11

due Thursday, April 30

Exercise 1. Show that every compact locally Euclidean space is second countable.

In particular, this applies to the spaces obtained by gluing polygon sides in pairs, and completes the proof that they are manifolds.

Exercise 2. Show that the fundamental group $\pi_1(X, x_0)$ is a group. More precisely, show that, for all loops α, β, γ at x_0,

a) $(\alpha * \beta) * \gamma \simeq \alpha * (\beta * \gamma)$,

b) $c * \alpha \simeq \alpha \simeq \alpha * c$,

c) $\alpha * \overline{\alpha} \simeq c \simeq \overline{\alpha} * \alpha$.

Here c is the constant loop $c(t) = x_0$ and $\overline{\alpha}(t) = \alpha(1-t)$.

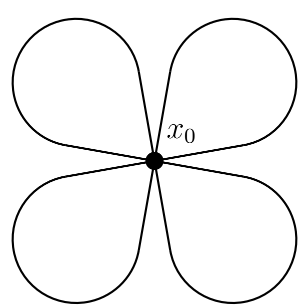
We already proved a) in class. Make sure you understand the proof and check the details.

Exercise 3. Let $x_0 \in S^1$ be a fixed point. Let

$$X = (S^1 \times \{1, \dots, n\})/\sim$$

with the equivalence relation defined by $(x_0, i) \sim (x_0, j)$ for all i, j (the finite set $\{1, \dots, n\}$ carries the discrete topology). It is called a *bouquet of circles* or *connected sum of circles*. Show that it is connected and Hausdorff, but not locally Euclidean if $n \geq 2$.

Hint: What happens if you remove a point?



Definition 1. A subset $A \subset \mathbb{R}^n$ is *convex* if for every two points $x, y \in A$ the line segment $\overline{xy}$ between them is contained in A.

A subset $A \subset \mathbb{R}^n$ is *star shaped* if there is a point $x_0 \in A$ such that for every point $x \in A$ the line segment $\overline{x_0 y}$ is contained in A.

Exercise 4.

a) Find a set which is star shaped, but not convex.

b) Show that $\pi_1(A, x_0)$ is the trivial group for every star shaped set $A \subset \mathbb{R}^n$.