Spring 2020 – Math 367K – 53115

Exercise Sheet 12

due Thursday, May 7

Exercise 1. Let X be a topological space and $\mathcal{U}$ an open cover of X. Let $\gamma\colon I \to X$ be a path. Show that there are real numbers $0 = t_0 < t_1 < \cdots < t_n = 1$ and $U_1, \ldots, U_n \in \mathcal{U}$ with

$$\gamma([t_{i-1}, t_i]) \in U_i \qquad \forall i \in \{1, \ldots, n\}.$$

Exercise 2. Assume $n \geq 2$. We consider the n–dimensional sphere

$$S^n = \Big\{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} x_i^2 = 1\Big\},$$

with the two special points $N = (1, 0, \ldots, 0)$ and $S = (-1, 0, \ldots, 0)$. Let $\gamma\colon I \to S^n$ be a loop with $\gamma(0) = \gamma(1) = N$.

a) Show that $S^n \setminus \{N\}$ and $S^n \setminus \{S\}$ are simply connected.

b) Show that there exists $k \in \mathbb{N}$ and real numbers $0 = t_0 < t_1 < \cdots < t_{2k+1} = 1$ such that, for all suitable $i \in \mathbb{N}$,

$$\gamma([t_{2i}, t_{2i+1}]) \in S^n \setminus \{S\}, \qquad \gamma([t_{2i-1}, t_{2i}]) \in S^n \setminus \{N\}.$$

That is, the sections alternatingly avoid either S or N.

c) Show that γ is homotopic to a loop which avoids S.

d) Show that γ is homotopic to a constant loop, and hence S^n is simply connected.

Exercise 3. Let X be a topological space, $A \subset X$ and $x_0 \in A$. A continuous map

$$h\colon I \times X \to X$$

such that

- $h(0, x) = x$ for all $x \in X$,
- $h(1, a) = a$ for all $a \in A$,

- $h(1, x) \in A$ for all $x \in X$

is called a *deformation retraction* of X onto A.

Show that if X admits a deformation retraction onto A, then $\pi_1(X, x_0) \cong \pi_1(A, x_0)$. Use this to show that $\pi_1(\mathbb{R}^2 \setminus \{0\}) \cong \pi_1(S^1)$.

Exercise 4. Let X, Y be topological spaces, $x_0 \in X$, $y_0 \in Y$, and

$$p_1\colon X \times Y \to X, \qquad p_2\colon X \times Y \to Y$$

the projections. Show that the map

$$p_{1*} \times p_{2*}\colon \pi_1(X \times Y, (x_0, y_0)) \to \pi_1(X, x_0) \times \pi_1(Y, y_0)$$

is a group isomorphism, so $\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$.