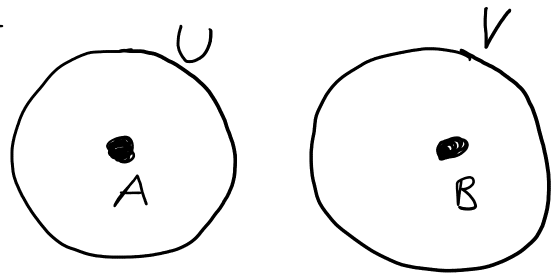
Before the break:

- metric space (X, d)
  $d : X \times X \longrightarrow \mathbb{R}^{\geq 0}$

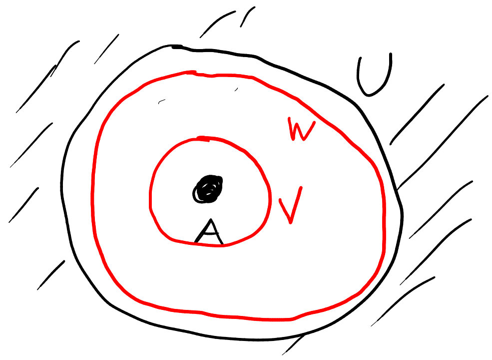
  topology generated by open ball
  $\hookrightarrow$ first countable
  $\hookrightarrow$ T_4



& one-point sets are closed

Alternatively

If X is T_4 and $A \subset U \subset X$ with A closed and U open, then $\exists\, V \subset X$ open such that $A \subset V$ & $\overline{V} \subset U$.



$X \setminus U = B$

$X \setminus \overline{V} = (X \setminus V)^{o} \supset W \supset X \setminus U$

Theorem (Urysohn metrization Theorem)
Any second countable T_3 space is metrizable.

We already proved:
Every second countable T_3 space is T_4.

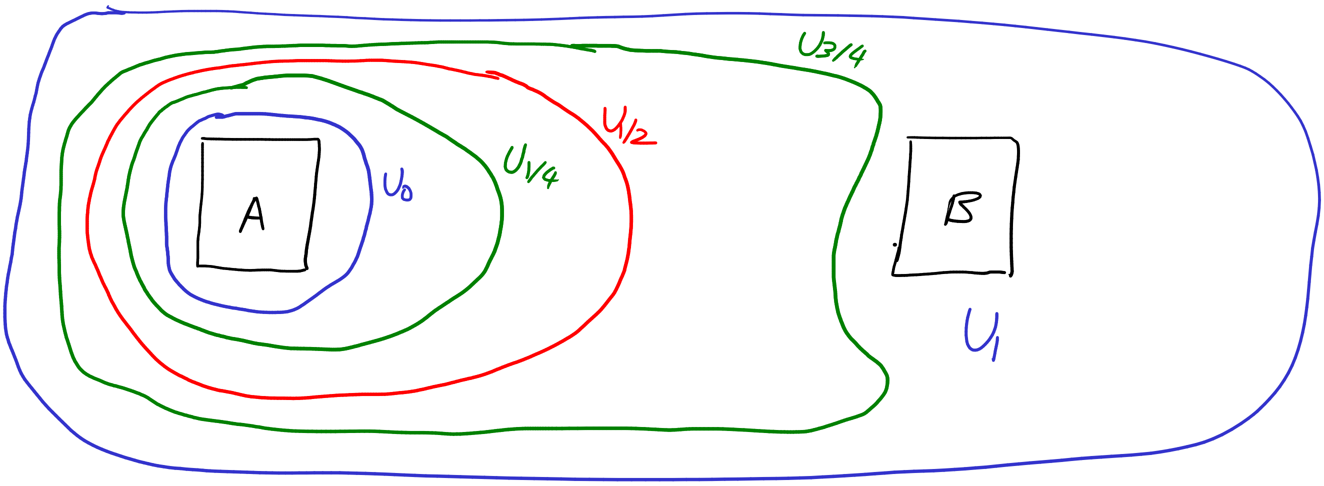
$$\left(\underset{d_X}{X} \times \underset{d_Y}{Y} \qquad d((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) + d_Y(y_1, y_2) \right)$$

__Theorem__ (Urysohn Lemma)

Let X be a T_4 space and $A, B \subset X$ disjoint non-empty closed subsets. Then there exists a continuous function $f: X \to I$ with $f(A) = \{0\}$ and $f(B) = \{1\}$.

__Proof__

We start by constructing a family U_p of open subsets of X for certain rational numbers p such that $\overline{U_p} \subset U_q$ whenever $p < q$. First set $U_1 = X \setminus B$. By T_4, we can find an open-set $U_0 \subset X$ with $A \subset U_0$ and $\overline{U_0} \subset U_1$.



Now assume that U_p is defined already for all p of the form $p = \frac{a}{2^n}$, $n \in \mathbb{N} \cup \{0\}$, $0 \le a \le 2^n$. We want to define U_p for $p = \frac{b}{2^{n+1}}$, $0 < b < 2^{n+1}$ and b odd. Since $\underbrace{\frac{b-1}{2^{n+1}}}_{=q} < \underbrace{\frac{b}{2^{n+1}}}_{=p} < \underbrace{\frac{b+1}{2^{n+1}}}_{=r}$ and U_q and U_r are already constructed, we want $\overline{U_q} \subset U_p$ and $\overline{U_p} \subset U_r$. We already know that $\overline{U_q} \subset U_r$, so by T_4 there exists such a U_p. The sets chosen in this way clearly satisfy $\overline{U_p} \subset U_q$ if $p < q$. Iterating this, we get U_p for all $p \in \mathbb{D} \cap [0,1]$ where

$$\mathbb{D} = \left\{ \frac{a}{2^n} \,\middle|\, a \in \mathbb{Z},\ n \in \mathbb{N} \right\}$$

are called the <u>dyadic numbers</u>. They are dense in $\mathbb{R}$.

For every $p \in \mathbb{D}$ which is not in $[0,1]$ we set

$$U_p = \emptyset \quad \text{if} \quad p < 0$$
$$U_p = X \quad \text{if} \quad p > 1.$$

It is still true that $\overline{U_p} \subset U_q$ if $p<q$.
Now we can define $f: X \longrightarrow I$ by

$$\mathbb{D}(x) = \{p \in \mathbb{D} \mid x \in U_p\}, \qquad f(x) = \inf \mathbb{D}(x).$$

Because $\mathbb{D} \cap (1,\infty) \subset \mathbb{D}(x) \subset [0,\infty)$, $f(x) \in I$. If $x \in A \subset U_0$, then $0 \in \mathbb{D}(x)$, so $f(x) = 0$. If $x \in B$, then $x \notin U_1$ and $x \notin U_p$ for all $p \leq 1$, so $\mathbb{D}(x) = (1,\infty) \cap \mathbb{D}$ and $f(x) = 1$.

It remains to show that f is continuous. We first prove:

(1) If $x \in \overline{U_p}$, then $f(x) \leq p$,

(2) If $x \notin U_p$, then $f(x) \geq p$.

<u>Proof of (1)</u>

If $x \in \overline{U_p}$, then $x \in U_q$ ($\Rightarrow q \in \mathbb{D}(x)$) for all $q \in \mathbb{D}$ with $q > p$. So $\mathbb{D} \cap (p,\infty) \subset \mathbb{D}(x)$ and $f(x) \leq p$, because the dyadic numbers are dense in $\mathbb{R}$.

<u>Proof of (2)</u>

If $x \notin U_p$, then $x \notin U_q$ for all $q \leq p$. So $\mathbb{D}(x) \subset (p,\infty)$ and $f(x) \geq p$.

Let $x_0 \in X$. We want to prove that f is continuous at x_0, i.e. that for every nbh V of $f(x_0)$ there is a nbh. U of x_0 with $f(U) \subset V$. Let $V = (c,d)$ be a neighborhood of $f(x_0)$. Then we can find dyadic numbers p, q with $c < p < f(x_0) < q < d$. Set $U = U_q \setminus \overline{U_p}$. By (1) and (2), $x_0 \in U$. Furthermore if $x \in U$, then $x \in U_q \subset \overline{U_q}$, so $f(x) \leq q$, and $x \notin \overline{U_p}$, so $x \notin U_p$ and $f(x) \geq p$. So $f(U) \subset [p,q] \subset (c,d)$. Hence f is continuous at x_0. Since x_0 was arbitrary, f is continuous. $\square$