Still want to show Theorem 3.9, the Urysohn metrization Theorem, which says

Every second countable T_3 space is metrizable.

So far we proved:

- T_3 & second countable $\Rightarrow T_4$
- $T_4 \Rightarrow \forall A, B$ closed & disjoint $\exists f: X \to I$ with $f(A)=\{0\}$ and $f(B)=\{1\}$.

We will show the Urysohn metrization theorem by showing that a second countable T_3 space X is homeomorphic to a subset of the metric space $\mathbb{R}^{\mathbb{N}}$.

Recall:

- X^Y = set of functions $Y \to X$
- $X^{\mathbb{N}}$ = set of functions $\mathbb{N} \to X$
  = set of sequences in X

HW7 If (X,d) is a metric space then $X^{\mathbb{N}}$ is also a metric space with the metric D:

$$D((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{d(x_i,y_i)}{1+d(x_i,y_i)}$$

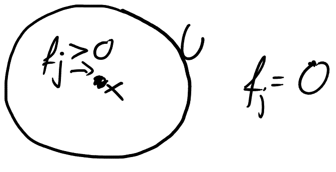
The metric topology is the product topology on $X^{\mathbb{N}}$.

Definition

A map $f: X \to Y$ between top. spaces X and Y is an imbedding if $f: X \to f(X)$ is a homeomorphism, where $f(X)$ is equipped with the subspace topology.

Theorem 3.11 (Imbedding Theorem)

Let X be top. space in which one element sets are closed. Let $(f_j)_{j\in J}$ be a family of continuous functions $f_j: X \to \mathbb{R}$. Assume that for every $x \in X$ and every neighborhood U of x there is a $j \in J$ such that $f_j(x) > 0$ and $f_j(X \setminus U) = \{0\}$.



Then the map

$$F: X \longrightarrow \mathbb{R}^J$$
$$x \longmapsto (f_j(x))_{j\in J}$$

is an imbedding.

Proof

By the definition of the product topology F is continuous if and only if $\pi_j \circ F$ is continuous for all $j \in J$. This true since $\pi_j \circ F = f_j$.
F is injective since, if $x \neq y$, then there is a function f_j with $f_j(x) > 0$ and $f_j(y) = 0$. But this means $F(x) \neq F(y)$.

$x \in U = X \setminus \{y\}$

x $\quad$ y

Let $Z = F(X)$. To show that F is an imbedding, we need to show that $F(U) \subset Z$ is open for every open set $U \subset X$. Let $z_0 \in F(U)$. Then $z_0 = F(x_0)$ for some $x_0 \in U$. By assumption there is a f_j with $f_j(x_0) > 0$ and $f_j(X \setminus U) = \{0\}$.
Set

$$W = Z \cap \pi_j^{-1}((0,\infty)) \subset \mathbb{R}^J. \qquad (W \text{ is open in } Z)$$

We claim that $z_0 \in W \subset F(U) \Rightarrow F(U)$ is open in Z.
First, $z_0 \in W$ since $\pi_j(z_0) = \pi_j(F(x_0)) = f_j(x_0) > 0$. Second, if $z \in W$ there is $x \in X$ with $z = F(x)$. Also $f_j(x) = \pi_j(F(x)) = \pi_j(z) > 0$, so $x \in U$. Hence $z \in F(U)$, so $W \subset F(U)$. Therefore $F(U)$ is open in Z and F is an imbedding. $\square$

Proof of Theorem 3.9 (Urysohn metrization Theorem)

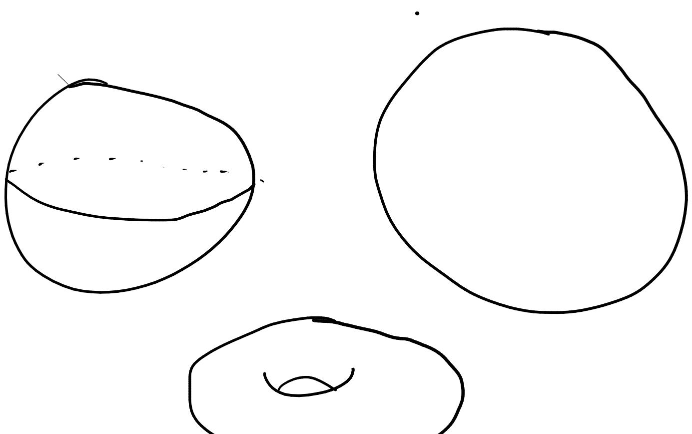
Let X be a second countable T_3 space. By Theorem 3.8 X is T_4.
Let $(B_i)_{i\in\mathbb{N}}$ be a countable basis for X. For every pair $(n,m) \in \mathbb{N}\times\mathbb{N}$ with $\overline{B_n} \subset B_m$ let $f_{n,m}: X \to \mathbb{R}$ be a continuous function with

$$f_{n,m}(\overline{B_n}) = \{1\} \text{ and } f_{n,m}(X \setminus B_m) = \{0\}.$$

Such a function exists by the Urysohn Lemma, Theorem 3.10. The set of these functions is countable and satisfies the requirements of Theorem 3.11: let $x \in X$ and U a neighborhood of x. Then there is a basis element B_m with $x \in B_m \subset U$. By T_4 there is an open set $V \subset X$ with $x \in V$ and $\overline{V} \subset B_m$. There is another basis element B_n with $x \in B_n \subset V$. So

$$x \in B_n \subset \overline{B_n} \subset \overline{V} \subset B_m \subset U \subset X.$$

Hence $f_{n,m}(x) = 1$ and $f_{n,m}(X \setminus U) \subset f_{n,m}(X \setminus B_m) = \{0\}$.
So by Theorem 3.11 there an imbedding from X to $\mathbb{R}^{\mathbb{N}}$. By HW7 $\mathbb{R}^{\mathbb{N}}$ is metrizable. So X is homeomorphic to a subset of a metrizable space, hence X is metrizable. $\square$



- locally homeo to $\mathbb{R}^n$, but not homeomorphic : <u>manifolds</u>