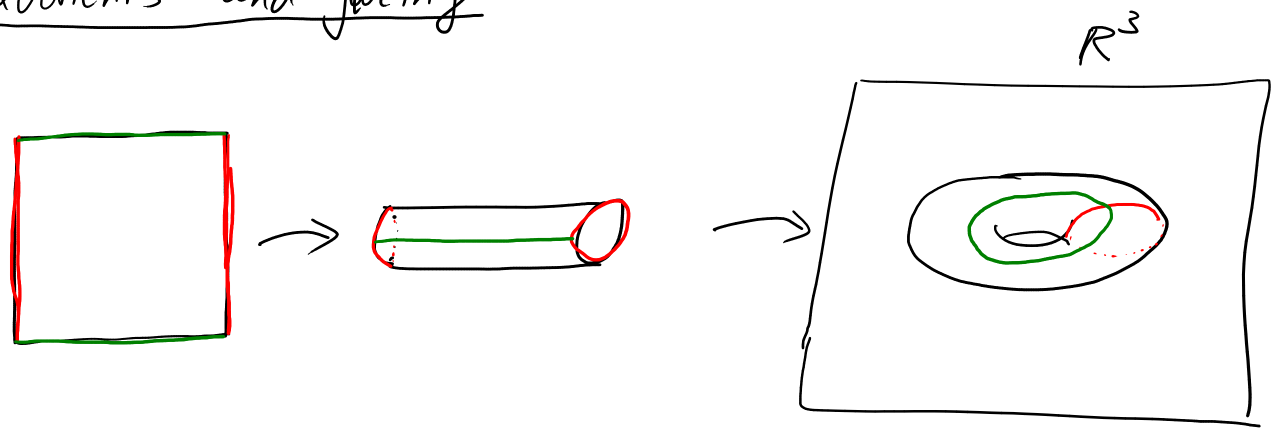
4. Quotients and glueing



Definition

Let X be a set. A <u>relation</u> R is a subset $R \subset X \times X$.
We write xRy for $(x,y) \in R$.
A relation $\sim$ is an <u>equivalence relation</u> if $\forall x,y,z \in X$

(1) $x \sim x$ (reflexive)

(2) $x \sim y \Rightarrow y \sim x$ (symmetric)

(3) $x \sim y$ & $y \sim z \Rightarrow x \sim z$ (transitive)

If $\sim$ is an equivalence relation and $x \in X$, then

$$x \in [x] = \{ y \in X \mid y \sim x \}$$

is called the equivalence class determined by x.

Lemma 4.1

Two equivalence classes for $\sim$ are either disjoint or equal.

Proof:

Let $[x]$, $[x']$ be two equivalence classes which are not disjoint. So there exists $y \in [x] \cap [x']$, i.e. $y \sim x$ and $y \sim x'$. By symmetry and transitivity $x \sim x'$. Now if $z \in [x]$, then $z \sim x$ and $x \sim x'$, so by transitivity $z \in [x']$. So $[x] \subset [x']$ and by a similar argument $[x'] \subset [x]$, so they are equal. □

Definition Let

$$X/\sim \; = \{ [x] \mid x \in X \}$$

be the set of equivalence classes, also called the quotient of X by $\sim$.

Examples

(1) On $\mathbb{Z}$ define the relation

$$a \sim b \iff a-b \text{ is divisible by } 5 \quad (5 \mid a-b)$$

That's an equivalence relation.

(1) $5 \mid a-a$ ✓

(2) $5 \mid a-b \Rightarrow 5 \mid b-a$ ✓

(3) $5 \mid a-b$ & $5 \mid b-c \Rightarrow 5 \mid a-c$ ✓ (because $a-c=(a-b)+(b-c)$)

$$\mathbb{Z}/\sim \; = \{[x] \mid x \in \mathbb{Z}\} = \{[0],[1],[2],[3],[4]\}$$

$[0] = \{\dots,-10,-5,0,5,10,\dots\}$, $[4] = \{\dots,-6,-1,4,9,14,\dots\}$

(2) On $\mathbb{R}^n \setminus \{0\}$, define the relation

$$x \sim y \iff \exists \lambda \in \mathbb{R}\setminus\{0\} : y = \lambda x$$

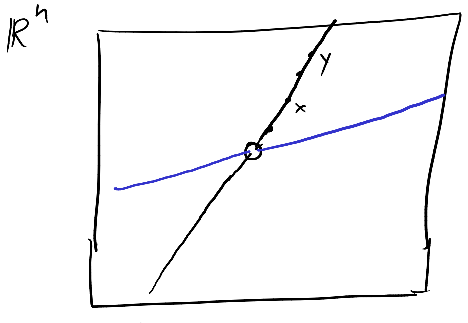
$\sim$ is an equivalence relation:

(1) $x = 1 \cdot x$

(2) $y = \lambda x \Rightarrow x = \lambda^{-1} y$

(3) $y = \lambda x$ & $z = \mu y \Rightarrow z = \lambda\mu x$ $\lambda, \mu \in \mathbb{R}\setminus\{0\}$, $x,y,z \in \mathbb{R}^n \setminus\{0\}$

$(\mathbb{R}^n \setminus \{0\})/\sim$ = "set of all lines in $\mathbb{R}^n$ through the origin"



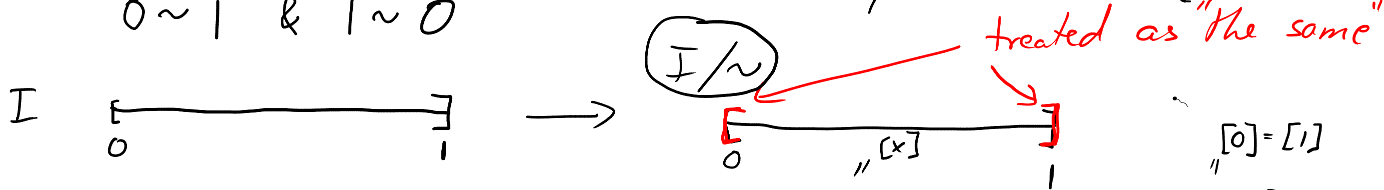
= real projective space $\mathbb{RP}^{n-1}$ of dimension $n-1$.

(3) Take $I = [0,1]$ and the relation

$x \sim x \quad \forall x \in I$

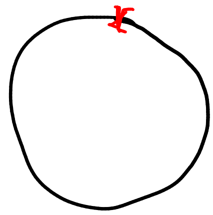
$0 \sim 1$ & $1 \sim 0$

$\Rightarrow$ this is an equivalence relation



$$I/\sim \; = \{\{x\} \mid x \in (0,1)\} \cup \{\{0,1\}\}$$

This "should be" S^1:



<u>Definition</u>

Let X be a topological space, Y a set and $p: X \to Y$ a surjective map. Then the <u>quotient topology</u> on Y is defined by

$Y = X/\sim$

$p: X \to X/\sim$, $x \mapsto [x]$

$$U \subset Y \text{ is open} \iff p^{-1}(U) \text{ is open.}$$

<u>Remark:</u>

The quotient topology is the <u>finest</u> topology on Y which makes p continuous.

<u>Examples</u>

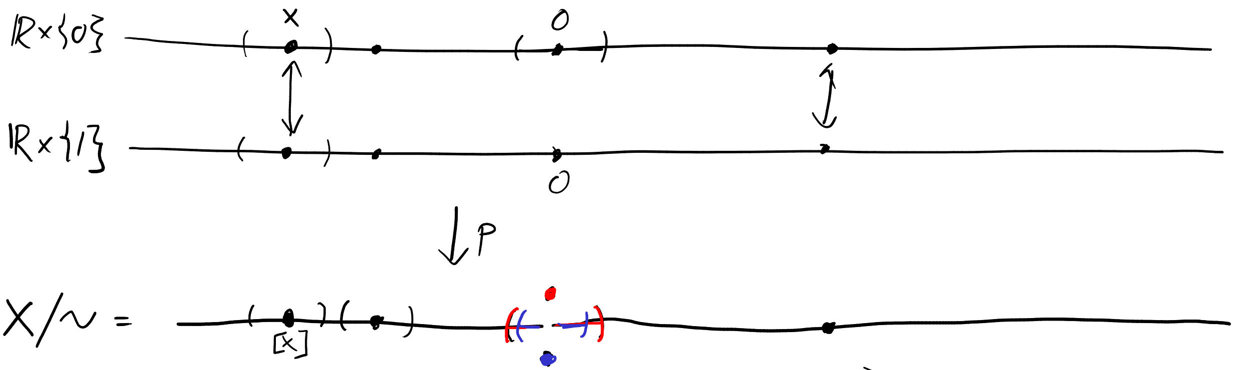
(1) I with $0 \sim 1$ $\Rightarrow$ $I/\sim$ with the quotient topology is homeomorphic to S^1

homeomorphism: $I/\sim \longrightarrow S^1 = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2=1\}$

$[t] \longmapsto (\cos(2\pi t), \sin(2\pi t))$ $\Rightarrow$ homework

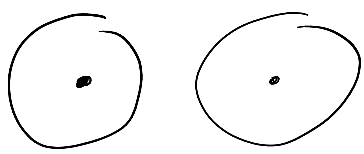
(2) $X = \mathbb{R} \sqcup \mathbb{R} = \mathbb{R} \times \{0,1\}$

$(x,0) \sim (x,1)$ $\forall x \in \mathbb{R} \setminus \{0\}$



"line with two origins"

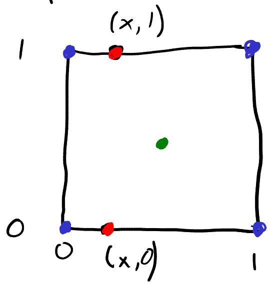
The topology on $X/\sim$ is not Hausdorff:



the two origins can not be separated by open sets.

(3) $X = I \times I$ $\quad (x,0) \sim (x,1) \quad \forall x \in I$

$(0,x) \sim (1,x) \quad \forall x \in I$



The 4 blue points are all in one equivalence class

Opposite points on sides are in the same eq. class

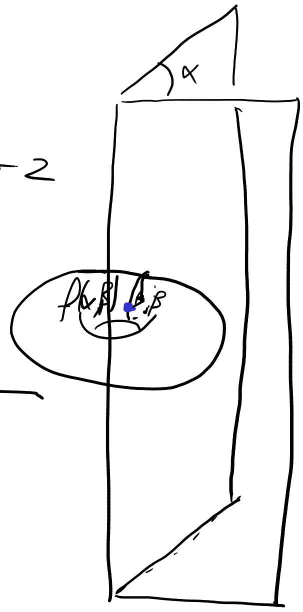
A point in the interior is an equivalence class on its own.

$X/\sim$ is homeomorphic to the torus T^2

$$X/\sim \xrightarrow{\approx} I/\sim \times I/\sim \xrightarrow{\approx} S^1 \times S^1 \xrightarrow{f} T^2$$

from ex. (1)

$$[(x,y)] \longmapsto ([x],[y]) \qquad (\alpha,\beta) \longmapsto$$



$X = I \times I$ $\qquad (0,x) \sim (1, 1-x)$