<u>Recall:</u> 4. Quotients and gluing

If $\sim$ is an equivalence relation on a top. space X, then

$$[x] = \{y \in X \mid y \sim x\} \qquad X/\sim = \{[x] \mid x \in X\}$$

natural projection $\longrightarrow$ $p: X \longrightarrow X/\sim$ surjective
$\quad x \longmapsto [x]$

Quotient topology: $U \subset X/\sim$ is open $\Leftrightarrow$ $\underset{\{x \in X \mid [x] \in U\}}{\underset{\parallel}{p^{-1}(U)}}$ is open

p is continuous by def of quotient topology ($U \subset X/\sim$ open $\Rightarrow p^{-1}(U)$ open)
But also: $p^{-1}(U)$ open $\Rightarrow$ U open

<u>Definition</u>

A surjective fn. $f: X \to Y$ between top. spaces X, Y is a <u>quotient map</u> <u>if</u> $U \subset Y$ is open iff $f^{-1}(U)$ is open.

The natural projection is a quotient map!

If $f: X \to Y$ is a quotient map, then Y is homeomorphic to $X/\sim$ with $\sim$ defined by $x \sim x' \Leftrightarrow f(x) = f(x')$.

<u>Warning:</u> It is not true for a quotient map f that $f(U)$ is open for every open set U.

Counterexample: $[0,1]/\sim$, $\sim$ defined by $0 \sim 1$

$$p: [0,1] \longrightarrow [0,1]/\sim$$

$U = [0, \frac{1}{2})$ is open in $[0,1]$, but $p(U)$ is not open, since

$$\begin{aligned} p^{-1}(p(U)) &= \{x \in [0,1] \mid p(x) \in p(U)\} \\ &= \{x \in [0,1] \mid \exists y \in U : p(x) = p(y)\} \\ &= \{x \in [0,1] \mid \exists y \in U : x \sim y\} = [0, \tfrac{1}{2}) \cup \{1\} \end{aligned}$$

is not open.

Lemma 4.2

Let X be a top. space, $\sim$ an equivalence relation on X and $p: X \to X/\sim$ the natural projection.
Let $f: X \to Y$ be a map to another top. space Y. Assume that $f(x) = f(x')$ whenever $x \sim x'$.
Then there exists a unique map $g: X/\sim \;\to Y$ such that $f = g \circ p$.
g is continuous if and only if f is continuous.

Proof:

Existence:

Let $z \in X/\sim$. Choose $x \in z$ (remember, z is an equivalence class) and set $g(z) = f(x)$. This defines a function g. Now if $x' \in X$, then $p(x') \in X/\sim$ so $g(p(x')) = f(x)$ for some $x \in \underset{[x']}{\underset{\|}{p(x')}}$.
But this means $x \sim x'$, so $g(p(x')) = f(x) = f(x')$.

Uniqueness:

If $g \circ p = f$, then $\forall x \in X$: $g([x]) = g(p(x)) = f(x)$. This determines g uniquely.

Continuity

If g is cont., then $f = g \circ p$ is continuous.
Conversely, let f be continuous, and let $U \subset Y$ be open.
Then $p^{-1}(g^{-1}(U)) = f^{-1}(U)$ $\left[(f \circ g)^{-1}(A) = g^{-1}(f^{-1}(A)) \right]$ and this open since f is continuous. So $g^{-1}(U)$ is open, and g is cont. $\square$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "p"'] & Y \\ X/\sim \arrow[ur, dashed, "g"'] & \end{tikzcd}$$

Definition

- Let G be a group and X a top. space. A <u>(left) action</u> of G on X is a map

$$G \times X \longrightarrow X$$
$$(g, x) \longmapsto g \cdot x$$

such that

neutral element in G

- $1_G \cdot x = x \quad \forall x \in X$
- $(gh) \cdot x = g \cdot (h \cdot x) \quad \forall g, h \in G, x \in X$

- The group action defines an equivalence relation: $\forall x, y \in X$

$$x \sim y \iff \exists g \in G : y = g \cdot x$$

The quotient $X/G = X/\sim$ is called the <u>quotient of X by the action of G</u>.

Why is $\sim$ an equivalence relation?

1) reflexive: $x = 1_G \cdot x \Rightarrow x \sim x$
2) symmetric: $x \sim y \Rightarrow \exists g : y = gx \Rightarrow \exists g : g^{-1}y = g^{-1}(gx) = (g^{-1}g)x = 1_G \cdot x = x$

$\Rightarrow \exists h \in G : x = hy \Rightarrow y \sim x$

3) transitive: $x \sim y \;\&\; y \sim z \Rightarrow \exists g, h \in G : y = gx, z = hy$

$\Rightarrow \quad (hg)x = h(gx) = hy = z \Rightarrow x \sim z$

<u>Example</u>:

Consider the following action of $(\mathbb{Z}, +)$ on $\mathbb{R}$:

$$\mathbb{Z} \times \mathbb{R} \longrightarrow \mathbb{R}$$
$$(z, r) \longmapsto z + r$$

This is a group action. The quotient $\mathbb{R}/\mathbb{Z} = \{[x] \mid x \in \mathbb{R}\}$

$[x] = \{\dots, x-3, x-2, x-1, x, x+1, x+2\}$

Claim: $\mathbb{R}/\mathbb{Z} \cong S^1$

To show this, define

$$\tilde{f}: \mathbb{R} \longrightarrow S^1 = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2=1\}$$
$$t \longmapsto (\cos(2\pi t), \sin(2\pi t))$$

$\tilde{f}$ is continuous, and invariant by the $\mathbb{Z}$-action:

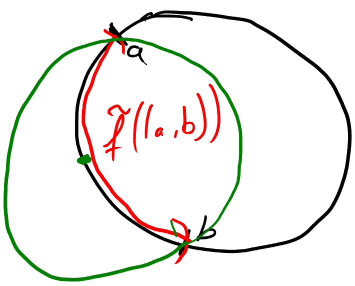
$$\forall t \in \mathbb{R}, z \in \mathbb{Z}: \quad \tilde{f}(t+z) = (\cos(2\pi t + 2\pi z), \sin(2\pi t + 2\pi z))$$
$$= (\cos(2\pi t), \sin(2\pi t)) = \tilde{f}(t)$$

$\Rightarrow$ By Lemma 4.2 there is a unique cont. function

$$f: \mathbb{R}/\mathbb{Z} \longrightarrow S^1 \qquad \boxed{\tilde{f} = f \circ p} \qquad p: \mathbb{R} \to \mathbb{R}/\mathbb{Z}$$
$$[x] \longmapsto (\cos(2\pi x), \sin(2\pi x))$$

- f is injective, since $\forall t,s \in \mathbb{R}$: $\left.\begin{matrix}\cos(2\pi t) = \cos(2\pi s) \\ \sin(2\pi t) = \sin(2\pi s)\end{matrix}\right\} \Rightarrow s - t \in \mathbb{Z}$
  $\Rightarrow s \sim t \Rightarrow [s] = [t] \in \mathbb{R}/\mathbb{Z}$
- f is surjective, since $\forall p \in S^1$: $p = (\cos(2\pi t), \sin(2\pi t))$ for some $t \in \mathbb{R}$. So $f([t]) = p$.
- $\tilde{f}$ maps open sets to open sets: Only need to show this for open intervals $(a,b) \subset \mathbb{R}$.

$$\tilde{f}((a,b)) = S^1 \cap B_{|\tilde{f}(b) - \tilde{f}(\frac{a+b}{2})|}\left(\tilde{f}\left(\frac{a+b}{2}\right)\right) \qquad (\text{if } b-a<1)$$



- f homeomorphism: If $U \subset \mathbb{R}/\mathbb{Z}$ is open, then $p^{-1}(U) \subset \mathbb{R}$ is open, so $\tilde{f}(p^{-1}(U))$ is open.

$$f(\underbrace{p(p^{-1}(U))}_{U}) = f(U)$$

$p(p^{-1}(U)) = U$

$p^{-1}(p(U)) =$ everything equivalent to something in U

$$\Rightarrow \quad \mathbb{R}/\mathbb{Z} \cong S^1$$

$I/\sim \qquad 0 \sim 1$

$\cong$

$\mathbb{R}/\mathbb{Z}$

HW

$I = [0,1]$ compact

$\mathbb{R} \cong (0,1)$ not compact