## Remember

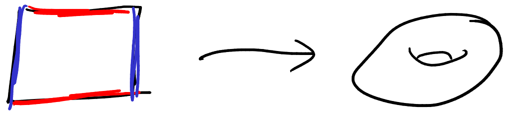
$X$ top. space, equivalence relation $\sim$ on $X$

$\longrightarrow$ surjective map $p : X \longrightarrow X/\sim$ natural/canonical projection

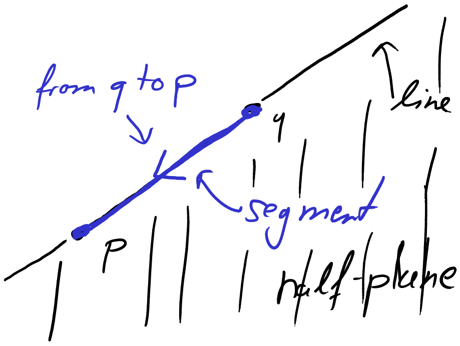
$\longrightarrow$ topology on $X/\sim$: $U \subset X/\sim$ open $\Leftrightarrow$ $p^{-1}(U)$ is open

Application 1: group action

Application 2: Gluing polygon

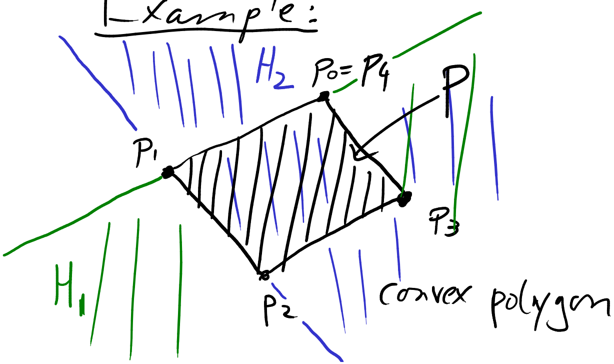


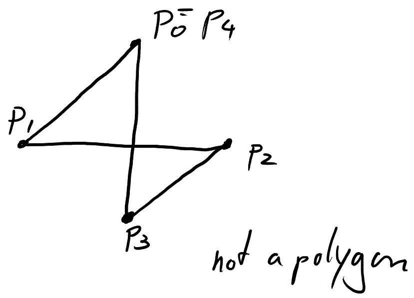
Let $p, q \in \mathbb{R}^2$, $p \neq q$. Then

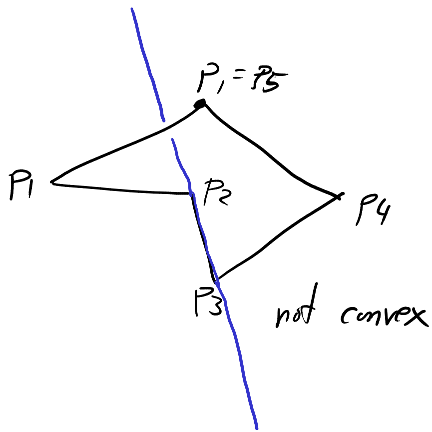



- the line through $p$ and $q$ is the set $\{p + t(q-p) \mid t \in \mathbb{R}\} \subset \mathbb{R}^2$
- the segment between $p$ and $q$ is the set $\overline{pq} = \{p + t(q-p) \mid t \in [0,1]\}$
- an orientation on the segment $\overline{pq}$ is a choice or order of its endpoint, i.e. one of "from $p$ to $q$" or "from $q$ to $p$".
- the (open) half-planes determined by the line through $p$ and $q$ are the connected components of $\mathbb{R}^2 \setminus \{p + t(q-p) \mid t \in \mathbb{R}\}$
- Let $p_0, p_1, \dots, p_n \in \mathbb{R}^2$ with $p_0 = p_n$, $n \geq 3$, be pairwise distinct points such that the segments $\overline{p_{i-1}p_i}$ and $\overline{p_{j-1}p_j}$ don't intesect except possibly at their endpoints, for all $i, j \in \{1, \dots, n\}$.
  Then these point define a polygon in the plane.
- the polygon defined by $p_0, \dots, p_n$ is convex if, for all $i \in \{1, \dots, n\}$, there is a half-plane $H_i$ defined by the points $p_{i-1}$ and $p_i$ such that all other points $p_k$ are contained in $H_i$.

Example:



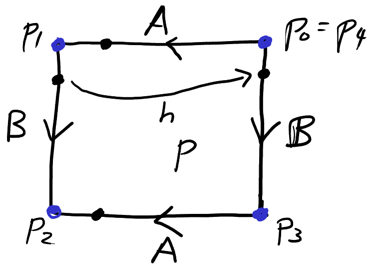
If $p_0, \dots, p_n$ define a convex polygon, the <u>polygonal region</u> is

$$P = \overline{H_1} \cap \overline{H_2} \cap \dots \cap \overline{H_n}$$
$$P^o = H_1 \cap H_2 \cap \dots \cap H_n$$
$$\partial P = \bigcup_{i=1}^{n} \overline{p_{i-1}p_i}$$

- A <u>labelling</u> is a map from the set of edges of a polygon to some set of <u>labels</u>.
- Given two oriented segments, $L$ from $a$ to $b$ and $L'$ from $c$ to $d$, the <u>positive linear map</u> from $L$ to $L'$ is the map

$$h: L \to L', \quad h(a+t(b-a)) = c+t(d-c) \quad \forall t \in [0,1]$$

- Given a convex polygon $P$ with labeled and oriented edges, define an equivalence relation in the following way:
whenever edges $L, L'$ have the same label, then let $h: L \to L'$ be the positive linear map and set $x \sim h(x) \quad \forall x \in L$
($\text{also } x \sim x \quad \forall x \in P$)




This gives us a quotient space $P/\sim$.

<u>This is actually an equivalence relation:</u>

- <u>symmetry</u>: the positive linear map from $L'$ to $L$ is inverse to the positive linear map from $L$ to $L'$

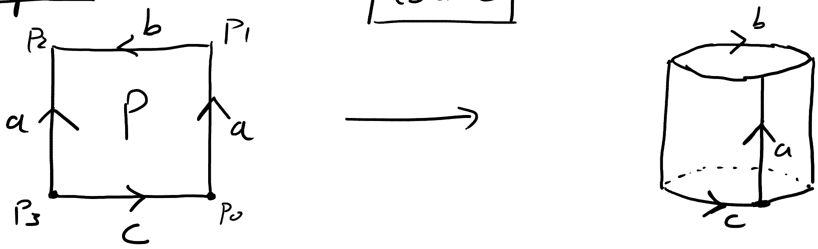
$$x \sim h(x) \ \forall x \in L, \quad y \sim h^{-1}(y) \ \forall y \in L'$$
$$\Downarrow$$
$$h(x) \sim x \quad \forall x \in L$$

- <u>transitivity</u>: Let $L, L', L''$ be oriented segments and $h_1: L \to L'$ and $h_2: L' \to L''$ the positive linear maps. Then the positive linear map from $L$ to $L''$ is $h_2 \circ h_1$.
For vertices $p_i$ we need to possibly add some more equivalences to make $\sim$ transitive.

## Examples:

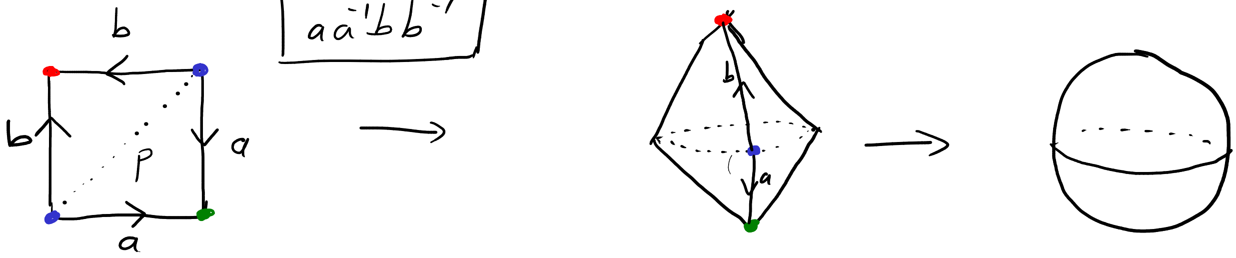
$\boxed{aba^{-1}c}$

(1)



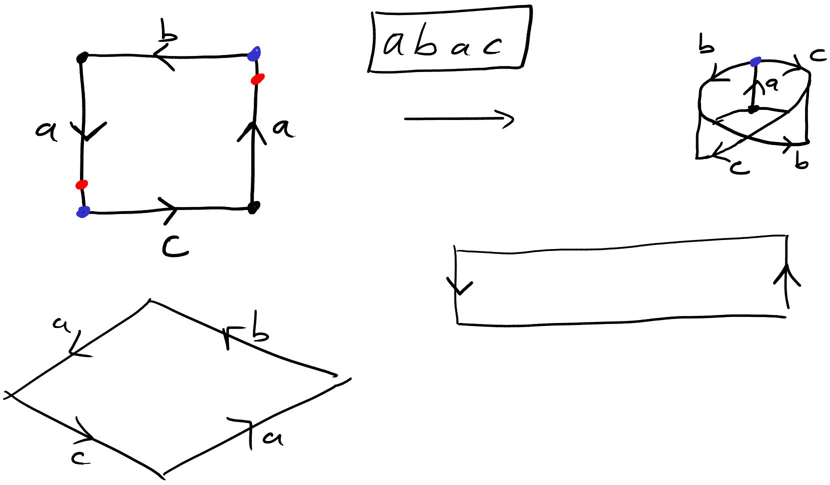

The quotient $P/\sim$ is homeomorphic to a cylinder

(2) $\boxed{aa^{-1}bb^{-1}}$




$$P/\sim \;\cong\; S^2$$

(3) $\boxed{abac}$




Möbius strip

- Labelling scheme: For $i \in \{1, \ldots, n\}$ let $\varepsilon_i$ be $+1$ if the edge $\overline{p_{i-1}p_i}$ is oriented from $p_{i-1}$ to $p_i$ and let $\varepsilon_i = -1$ if $\overline{p_{i-1}p_i}$ is oriented from $p_i$ to $p_{i-1}$. Let $a_i$ be the label of the edge $\overline{p_{i-1}p_i}$. Then the symbol

$$a_1^{\varepsilon_1} a_2^{\varepsilon_2} \cdots a_n^{\varepsilon_n}$$

is called a labelling scheme.

## Theorem 4.3

Let $P$ and $P'$ be two convex polygons with oriented and labeled edges. If their labeling schemes agree, then the quotients $P/\sim$ and $P'/\sim$ are homeomorphic.