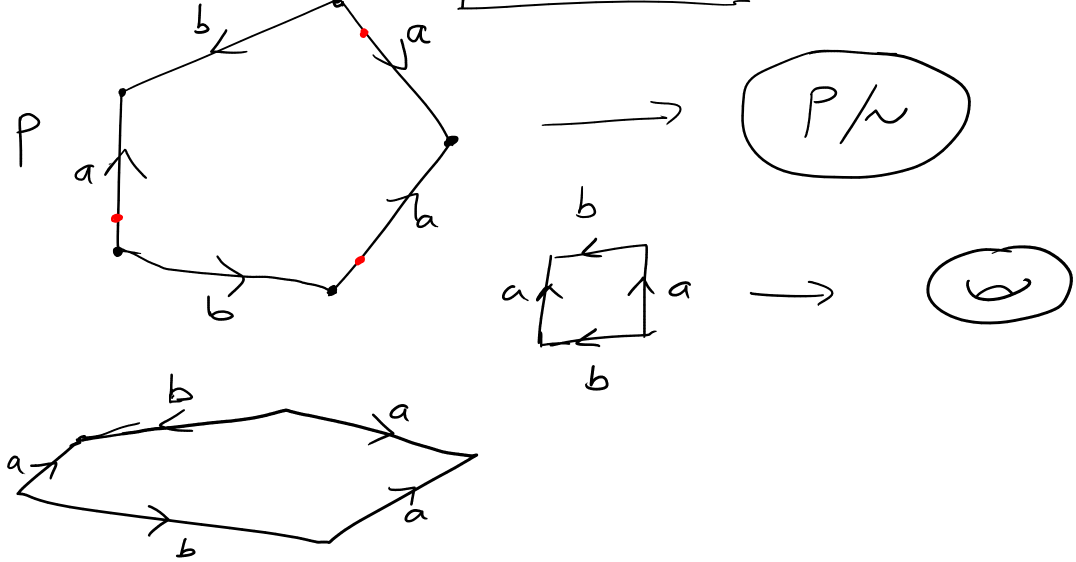
Last time:

$\boxed{a a^{-1} b a^{-1} b}$



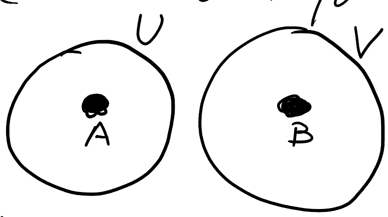
Theorem 4.4 $p: P \longrightarrow P/\sim$

Let $X = P/\sim$ be the quotient of a gluing of the polygon P. Then X is compact and Hausdorff (and even T_4).

Lemma 4.5

Let $p: X \to Y$ be a closed quotient map. If X is T_4, then Y is T_4.

Proof:



+ one-point sets are closed.

Every one-point set in X is closed and p maps them to one-point sets in Y, which are also closed.

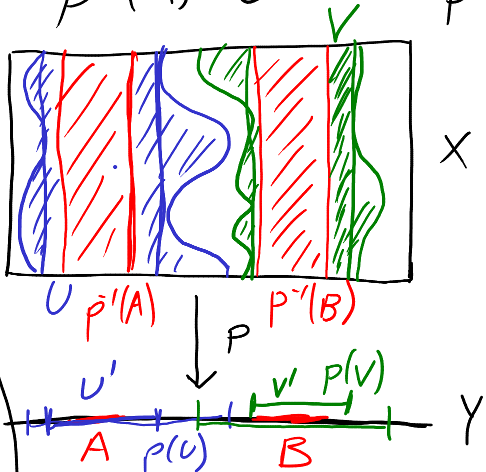
Let $A, B \subset Y$ be disjoint non-empty and closed. Then $p^{-1}(A)$ and $p^{-1}(B)$ are disjoint, non-empty closed sets in X, so there exist $U, V \subset X$ open, disjoint such that $p^{-1}(A) \subset U$ and $p^{-1}(B) \subset V$.

[if $x \in p^{-1}(A) \cap p^{-1}(B)$, then $p(x) \in A$, $p(x) \in B$, so $A \cap B \neq \emptyset$]

C "saturated" if $x \in C \Rightarrow y \in C$, $p(x) = p(y)$

fiber $p^{-1}(\{y\})$

saturated = a union of fibers



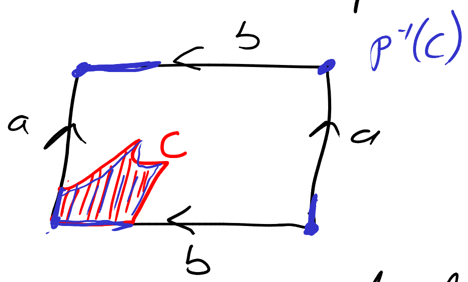
Let $U' = Y \setminus p(X\setminus U)$ and $V' = Y\setminus p(X\setminus V)$.

- They are open, because p maps closed sets to closed sets.
- If $y \in U' \cap V'$, then $y = p(x)$ for some $x \in X$, but $x \notin X \setminus U$ and $x \notin X\setminus V$, so $x \in U$ and $x \in V$, which is impossible since U, V are disjoint. So $U' \cap V' = \emptyset$
- If $y \in A$, but $y \notin U'$, then $y \in p(X\setminus U)$, so there exists $x \in X \setminus U$ with $y = p(x)$, so $x \in p^{-1}(A) \subset U$. This is a contradiction, so $A \subset U'$. Similarly $B \subset V'$. $\square$

<u>Proof of Theorem 4.4</u>

- $P/\sim$ is compact since it is the image $p(P)$ of the compact space P.
- Lemma 4.5 shows that $P/\sim$ is T_4 if $p: P \to P/\sim$ is closed. To see that let $C \subset P$ be a closed subset. Then $p^{-1}(p(C))$ is a union of:
  - some vertices of P
  - C
  - For every edge e and every other edge e' which has the same label, $C \cap e$ and $h(C\cap e)$, where $h: e \to e'$ is the pos. linear map.

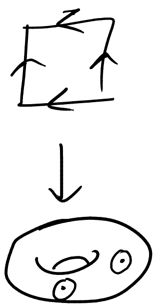
closed in P

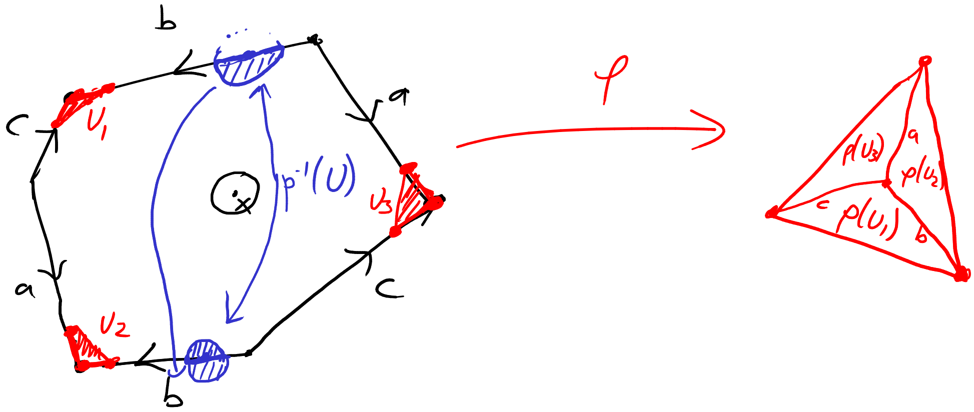


So $p^{-1}(p(C))$ is a union of finitely many closed sets, and therefore closed. So $p(C) \subset P/\sim$ is closed.
So p is a closed map and $P/\sim$ is T_4. $\square$

Theorem 4.6

Let $X = P/\sim$ be obtained from a polygon gluing, where each label occurs on exactly 2 edges. Then X is locally homeomorphic to $\mathbb{R}^2$, meaning $\forall x \in X$ there exists a neighborhood U of x and a homeomorphism $\varphi: U \to V$ to an open set $V \subset \mathbb{R}^2$.





Definition

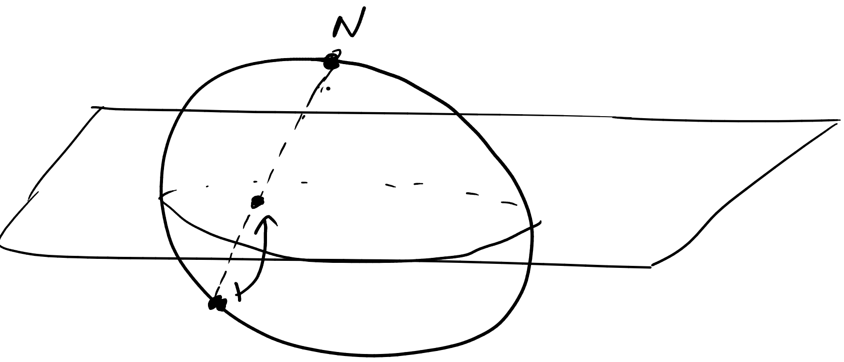
A top. space X is called a <u>manifold</u> if it is

- first countable
- Hausdorff
- locally Euclidean, meaning around every point $x \in X$ there is a neighborhood homeomorphic to an open subset of $\mathbb{R}^n$, for some n.

smooth manifolds: differential topology

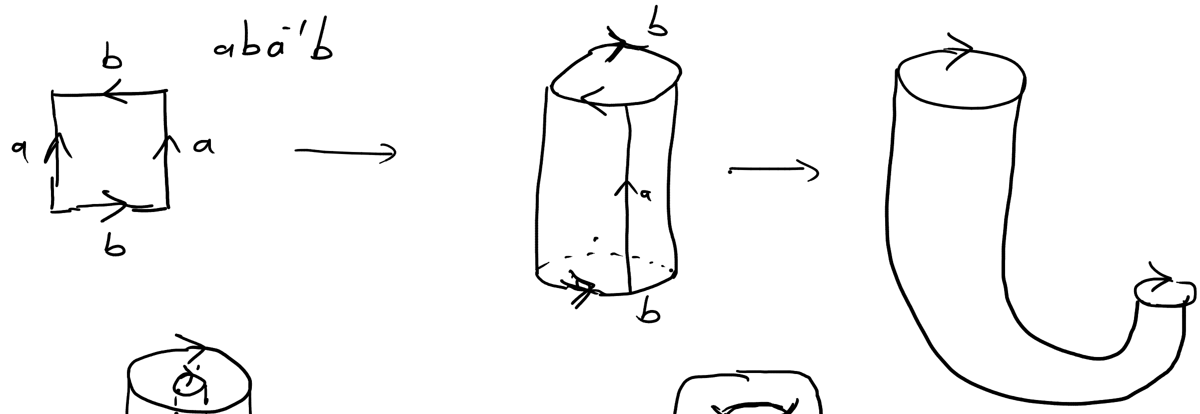
n is called the <u>dimension</u> of the manifold.

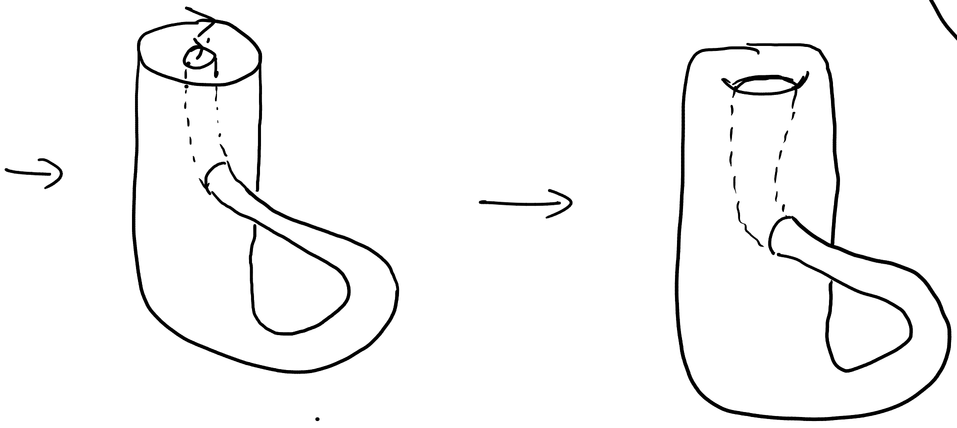
A <u>surface</u> is a 2-dimensional manifold.



$S^2 \setminus \{N\} \to \mathbb{R}^2$

homeomorphism





Klein bottle

This is a surface, but not imbeddable into $\mathbb{R}^3$
(the picture is "cheating" because it has a self-intersection)