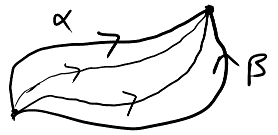
Remember

The fundamental group $x_0 \in X$

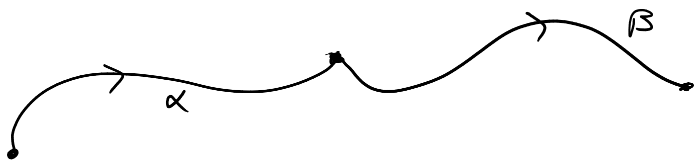
$$\pi_1(X, x_0) = \{ \gamma : I \to X \text{ loop at } x_0 \} / \simeq$$
$$\gamma(0) = \gamma(1) = x_0$$

$\alpha \simeq \beta$ if $\exists\, h : I \times I \to X$, continuous

$$h(0,t) = \alpha(t)$$
$$h(1,t) = \beta(t)$$
$$h(s,0) = \alpha(0) = \beta(0) = x_0$$
$$h(s,1) = \alpha(1) = \beta(1) = x_0$$



$\alpha * \beta$

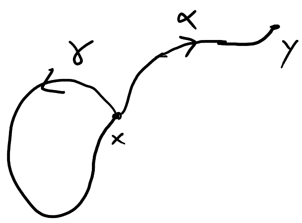


$(\pi_1(X, x_0), *)$ is a group.

Theorem 5.3

If X is path connected and $x, y \in X$, then $\pi_1(X,x)$ and $\pi_1(X,y)$ are isomorphic groups. More precisely, let $\alpha : I \to X$ be a path from x to y. Then

$$\varphi : \pi_1(X,x) \longrightarrow \pi_1(X,y)$$
$$[\gamma] \longmapsto [\bar{\alpha} * \gamma * \alpha]$$



is an isomorphism. Here $\bar{\alpha}$ is α traversed in the opposite direction, i.e. $\bar{\alpha}(t) = \alpha(1-t)$.

Proof:

constant path at x

$$\varphi([\gamma] * [\delta]) = \varphi([\gamma * \delta]) = [\bar{\alpha} * \gamma * \delta * \alpha] = [\bar{\alpha} * \gamma * c_x * \delta * \alpha]$$
$$= [\bar{\alpha} * \gamma * \alpha * \bar{\alpha} * \delta * \alpha] = \varphi([\gamma]) * \varphi([\delta]),$$

so φ is a group homomorphism. The map

$$\psi : \pi_1(X,y) \to \pi_1(X,x), \quad [\gamma] \longmapsto [\alpha * \gamma * \bar{\alpha}]$$

is also a group hom. and inverse to φ:

$$\varphi(\psi([\gamma])) = [\bar{\alpha} * \alpha * \gamma * \bar{\alpha} * \alpha] = [\gamma] \text{ and } \psi(\varphi([\gamma])) = [\alpha * \bar{\alpha} * \gamma * \alpha * \bar{\alpha}] = [\gamma]$$

$\square$

Definition

If X is path connected, $\pi_1(X, x_0)$ is called the fundamental group of X (regardless of the base point).

Theorem 5.4

If $f: X \to Y$ is continuous, then

$$\begin{aligned} f_* : \pi_1(X, x_0) &\longrightarrow \pi(Y, f(x_0)) \\ [\gamma] &\longmapsto [f \circ \gamma] \end{aligned}$$

is a well-defined group homomorphism, and satisfies $(f \circ g)_* = f_* \circ g_*$. for cont. maps $f: X \to Y$ and $g: Z \to X$.

Proof:

To see that f_* is well-defined, take loops γ, δ which are homotopic by a homotopy h. Then $f \circ h$ is a homotopy from $f \circ \gamma$ to $f \circ \delta$. So f_* is well-defined. Furthermore,

$$f \circ (\alpha * \beta) = (f \circ \alpha) * (f \circ \beta)$$

for any two paths α, β, so f_* is a group homomorphism. $\square$

Corollary 5.5

If $f: X \to Y$ is a homeomorphism, then $f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ is an isomorphism of groups.

Proof:

If f is a homeo, f has a continuous inverse f^{-1}. Consider

$$f_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, f(x_0))$$
$$(f^{-1})_* : \pi_1(Y, f(x_0)) \longrightarrow \pi_1(X, x_0)$$

By Theorem 5.4 both are group homomorphisms and

$$f_* \circ (f^{-1})_* = (f \circ f^{-1})_* = (id_Y)_* = id_{\pi_1(Y, f(x_0))}$$
$$(f^{-1})_* \circ f_* = (f^{-1} \circ f)_* = (id_X)_* = id_{\pi_1(X, x_0)}$$

So f_* is an isomorphism. $\square$

$\boxed{\text{So we can use the fundamental group to distinguish topological spaces. If they are not isomorphic, the spaces are not homeomorphic.}}$

$\mathbb{R}$, $\mathbb{R}^2$ have the same fundamental group (trivial group), but they are not homeomorphic (removing a pt. disconnects $\mathbb{R}$ but not $\mathbb{R}^2$).

From now on, we assume our spaces are path connected.

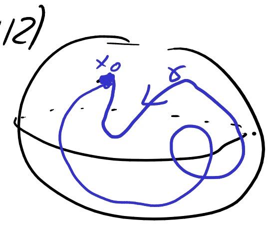
$\pi_1(X, x_0)$ only describes the path component containing x_0.

<u>Definition</u>

Let X be path connected, X is called <u>simply connected</u> if $\pi_1(X, x_0)$ is the trivial group.

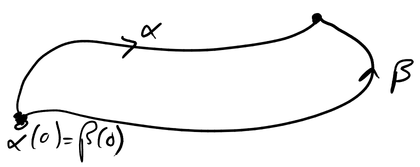
<u>Examples</u>:

- $\mathbb{R}^n$ is simply connected
- A star shaped set in $\mathbb{R}^n$ is simply connected. (HW 11)
- S^n is simply connected if $n \geq 2$. (HW12)



<u>Lemma 5.6</u>

If X is simply connected and $\alpha, \beta : I \to X$ are paths with $\alpha(0) = \beta(0)$ and $\alpha(1) = \beta(1)$, then $\alpha \simeq \beta$.

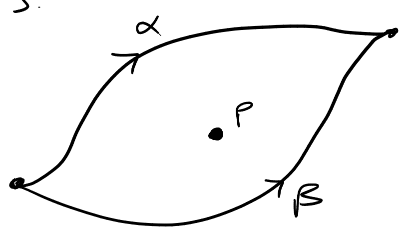


<u>Proof</u>.

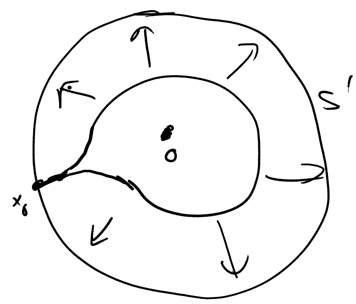
Consider $\alpha * \bar{\beta}$, This is a loop at $\alpha(0)$, so it is homotopic to the constant curve $c_{\alpha(0)}$. Hence

$$\beta \simeq c_{\alpha(0)} * \beta \simeq (\alpha * \bar{\beta}) * \beta \simeq \alpha * (\bar{\beta} * \beta) \simeq \alpha * c_{\alpha(1)} \simeq \alpha . \qquad \square$$

$\mathbb{R}^2 \setminus \{p\}$:



is not simply connected, so there are non-homotopic paths.



$\pi_1(\mathbb{R}^2 \setminus \{0\}) \cong \pi_1(S^1)$

$\cong \mathbb{Z}$

$$\mathbb{R}^2 / \mathbb{Z}^2 \cong T^2 \cong S^1 \times S^1$$

"easy" theorem (maybe homework?)

$\mathbb{R}, \mathbb{R}^2$ simply conn. $\quad \pi_1(T^2) = \pi_1(S^1 \times S^1) \cong \pi_1(S^1) \times \pi_1(S^1) \cong \mathbb{Z} \times \mathbb{Z}$

$\mathbb{R}/\mathbb{Z} \cong S^1 \qquad \pi_1(S^1) = \mathbb{Z}$

$\hookrightarrow$ covering spaces (also used for computation of $\pi_1(S^1)$)