Algebraic Topology: Using abstract algebra to study topological spaces

$\boxed{M382C}$

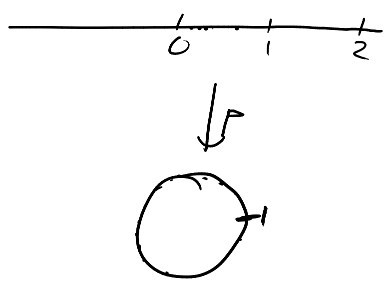
$\pi_1(X, x_0) = \{\gamma : I \to X \text{ loops at } x_0\}/\simeq$ fundamental group

Does not depend on basepoint (upto isomorphism), and is invariant under diffeomorphism.

e.g. $\pi_1(\mathbb{R}^n, x_0) = \{1\}$

Today: Fundamental group of S^1 (using covering space theory)

Let $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$, $p : \mathbb{R} \to S^1$.
$t \mapsto e^{2\pi i t}$



Choose the basepoint $0 \in \mathbb{R}$, $p(0) = 1 \in S^1$.

What is $\pi_1(S^1, 1)$?

Theorem 5.7

a) p has the path lifting property:
Given a point $x \in S^1$ and $y \in p^{-1}(x)$, and a path $\gamma : I \to S^1$ with starting point $\gamma(0) = x$, there exists a unique path $\gamma' : I \to \mathbb{R}$ starting at y such that $p \circ \gamma' = \gamma$. We call γ' the lift of γ with start pt. y.

b) p has the homotopy lifting property:
Given paths $\alpha, \beta : I \to S^1$ from x_0 to x_1 and a homotopy h from α to β, and a lift α' of α, then there exists a unique homotopy $h' : I \times I \to \mathbb{R}$ such that

$$h'(0, t) = \alpha'(t) \quad \text{and} \quad p \circ h' = h.$$

Proof

For every $z \in \mathbb{R}$, the map

$$p : (z, z+1) \longrightarrow S^1 \setminus \{p(z)\}$$

is a homeomorphism. (p is open & continuous). Let

$$s_z : S^1 \setminus \{p(z)\} \longrightarrow (z, z+1)$$

be its inverse.

a) There are real numbers $0 = t_0 < t_1 < \cdots < t_n = 1$ and $x_1, \dots, x_n \in S^1$ such that
$$\gamma([t_{i-1}, t_i]) \subset S^1 \setminus \{x_i\} \qquad \text{(homework 12)}$$
Inductively assume $\gamma'|_{[0,t_i]}$ is already constructed. Choose $z_{i+1} \in p^{-1}(\{x_{i+1}\})$ such that $\gamma'(t_i) \in (z_{i+1}, z_{i+1}+1)$. Then set
$$\gamma'|_{[t_i, t_{i+1}]} = s_{z_{i+1}} \circ \gamma|_{[t_i, t_{i+1}]}$$
This coincides with the old def. on t_i, so we have defined γ' on $[0, t_{i+1}]$. It clearly satisfies $\gamma = p \circ \gamma'$.

Uniqueness: If $\gamma', \gamma'' : I \to \mathbb{R}$ are two lifts of γ starting at $y \in \mathbb{R}$, then $\gamma'(t) - \gamma''(t) \in \mathbb{Z}$ for all $t \in I$, and since $\gamma' - \gamma''$ is cont., it is constant. But $\gamma'(0) = y = \gamma''(0)$, so $\gamma' = \gamma''$.

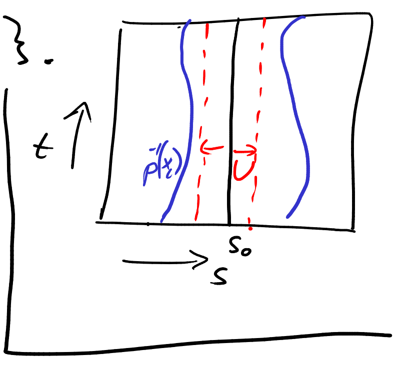
b) Uniqueness:

If h' is as required, then for any $t \in I$ the path $h'(-,t) : I \to \mathbb{R}$ is a lift of the path $h(-,t)$ in S^1 with starting point $h'(0,t) = \alpha'(t)$. By a) such a lift is unique, and shows uniqueness of h'.

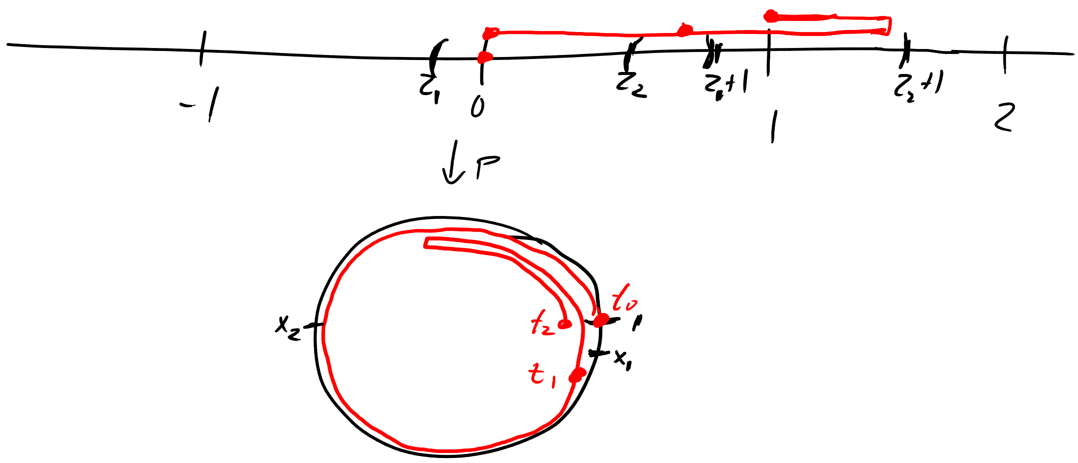
Existence:

By the above, h' exists as a map. We just have to show that it is continuous.

Let $s_0 \in I$. Then (by HW 12) there is a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ and $x_1, \dots, x_n \in S^1$
$$h(s_0, [t_{i-1}, t_i]) \subset S^1 \setminus \{x_i\}.$$
Since I is compact, there is a neighborhood U of s_0 in I such that
$$h(U \times [t_{i-1}, t_i]) \subset S^1 \setminus \{x_i\}.$$



Inductively, assume h' is cont. on $U \times [0, t_i]$. Then $h'(U \times \{t_i\}) \subset (z_{i+1}, z_{i+1}+1)$ for some $z_{i+1} \in p^{-1}(\{x_{i+1}\})$, and $h' = s_{z_{i+1}} \circ h$ on $U \times [t_i, t_{i+1}]$. So h' is cont. on $U \times [0, t_{i+1}]$. Since we chose s_0 arbitrarily, we get that h' is continuous everywhere. $\square$



If α and β are homotopic loops at $p(0) \in S^1$ and α', β' are their lifts with starting point 0, then $\alpha' \simeq \beta'$, so in particular $\alpha'(1) = \beta'(1)$. Hence we get a map

$$\varphi : \pi_1(S^1, 1) \longrightarrow \mathbb{Z} \qquad \text{"rotation number"}$$

$$[\gamma] \longmapsto \gamma'(1)$$

, where γ' is the lift of γ with starting point $\gamma'(0) = 0$.

__Theorem 5.8__

φ is an isomorphism, so $\pi_1(S^1, 1) \cong \mathbb{Z}$

__Proof:__

__Group homomorphism:__

Let α, β be loops in S^1 at 1, and α', β' their lifts with starting pt. 0. Let β'' be the lift of β with starting point $\alpha'(1)$. Then $\beta' - \beta''$ maps into $\mathbb{Z}$ and is constant. Furthermore, $\alpha' * \beta''$ is the lift of $\alpha * \beta$ starting at 0. So

$$\varphi([\alpha * \beta]) = (\alpha' * \beta'')(1) = \beta''(1) = \beta'(1) + (\beta'' - \beta')(1)$$
$$= \beta'(1) + (\beta'' - \beta')(0) = \beta'(1) + \alpha'(1) = \varphi([\alpha]) + \varphi([\beta]).$$

__Injective:__

If $\varphi([\alpha]) = \varphi([\beta])$, then $\alpha'(1) = \beta'(1)$ and $\alpha'(0) = \beta'(0) = 0$. Since $\mathbb{R}$ is simply connected α' and β' are homotopic by a homotopy h'. Then $p \circ h'$ is a homotopy from α to β, so $[\alpha] = [\beta]$.

__Surjective:__ Let $z \in \mathbb{Z}$. Let $\gamma' : I \to \mathbb{R}$ be defined by $\gamma'(t) = zt$, and $\gamma = p \circ \gamma'$. Then $\varphi([\gamma]) = \gamma'(1) = z$. $\square$