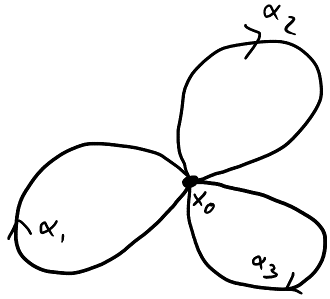
Last time:

Bouquet of n circles X



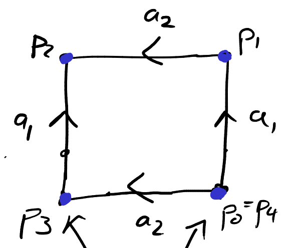
$$\langle a_1, \dots, a_n \rangle \xrightarrow{\sim} \pi_1(X, x_0)$$
$$a_i \longmapsto [\alpha_i]$$

Fundamental group is a free group

$$\langle a_1, \dots, a_n \rangle = \{1, a_1, a_2, \dots, a_1^{-1} a_2 a_3^4 a_2, \dots\}$$

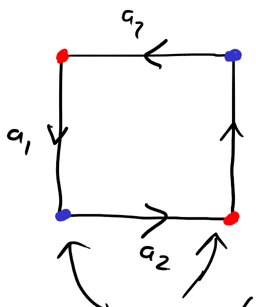
Today: Polygon P with edges labeled $a_1, \dots, a_n$ by the labelling scheme $a_{i_1}^{\varepsilon_1} a_{i_2}^{\varepsilon_2} \cdots a_{i_k}^{\varepsilon_k}$ $(\varepsilon_i \in \{\pm 1\})$.

$$\pi : P \longrightarrow P/\sim$$



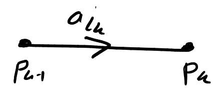
$a_1 a_2 a_1^{-1} a_2^{-1}$

a single vertex class



two vertex classes

if $\varepsilon_k = 1$

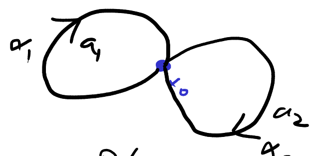


$\alpha_k(t) = \pi(t p_k + (1-t) p_{k-1})$

if $\varepsilon_k = -1$: $\alpha_k(t) = \pi(t p_{k-1} + (1-t) p_k)$

Assume that there is a single vertex class, i.e. all vertices of P get mapped to the same point in $P/\sim$ by π, call it x_0.

Then $\partial P/\sim$ is a bouquet of circles!
It has one circle for every label.

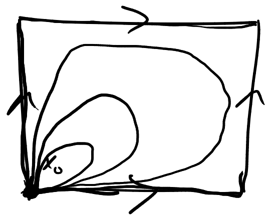


$\iota : \partial P/\sim \longrightarrow P/\sim$

$$\psi : \langle a_1, \dots, a_n \rangle \xrightarrow{\sim} \pi_1(\partial P/\sim, x_0) \xrightarrow{\iota_*} \pi_1(P/\sim, x_0)$$
$$a_i \longmapsto [\alpha_i]$$

Is ψ an isomorphism? No, it's not injective:

$$\psi\left(a_{i_1}^{\varepsilon_1} a_{i_2}^{\varepsilon_2} \cdots a_{i_k}^{\varepsilon_k}\right) = [c_{x_0}]$$



$a_{i_1}^{\varepsilon_1} \cdots a_{i_k}^{\varepsilon_k} \in \ker \psi$.

The kernel of a group homomorphism is a normal subgroup, so $N(a_{i_1}^{\varepsilon_1}\cdots a_{i_n}^{\varepsilon_n}) \subset \ker \varphi$

$\uparrow$ smallest normal subgroup of $\langle a_1,\dots,a_n\rangle$ containing $a_{i_1}^{\varepsilon_1}\cdots a_{i_n}^{\varepsilon_n}$.

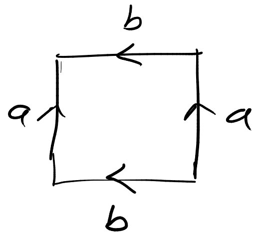
So we get a group homomorphism

$$\underset{\displaystyle \langle a_1,\dots,a_n \mid a_{i_1}^{\varepsilon_1}\cdots a_{i_n}^{\varepsilon_n} = 1\rangle}{\underset{\shortparallel}{\langle a_1,\dots a_n\rangle / N(a_{i_1}^{\varepsilon_1}\cdots a_{i_n}^{\varepsilon_n})}} \longrightarrow \pi_1(P/\sim, x_0)$$

One can show (similarly to Theorem 5.9) that this is an isomorphism.

Examples

1) torus

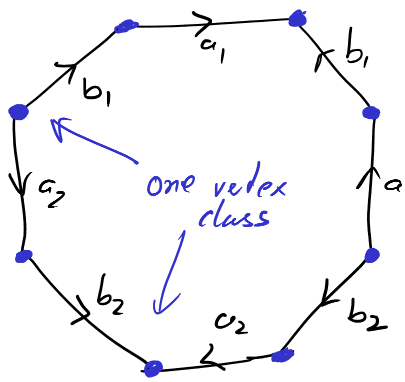


$\longrightarrow \pi_1 = \langle a,b \mid aba^{-1}b^{-1}=1\rangle$
$= \{a^x b^y \mid x,y\in\mathbb{Z}\} \cong \mathbb{Z}\times\mathbb{Z}$

2) Genus g surface $(g\geq 1)$

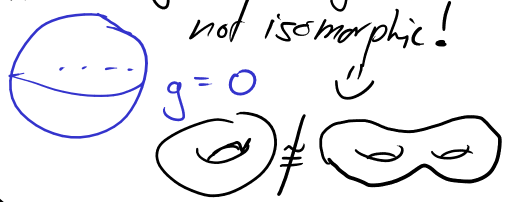
Labels: $a_1,\dots,a_g, b_1,\dots,b_g$, labelling scheme $\prod_{i=1}^{g} a_i b_i a_i^{-1} b_i^{-1}$

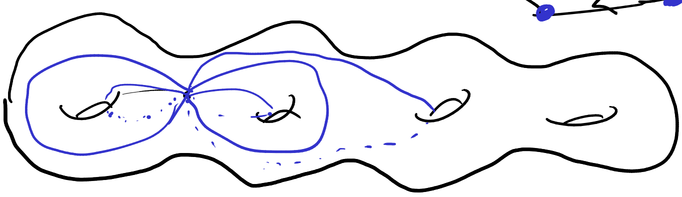
$g=2$

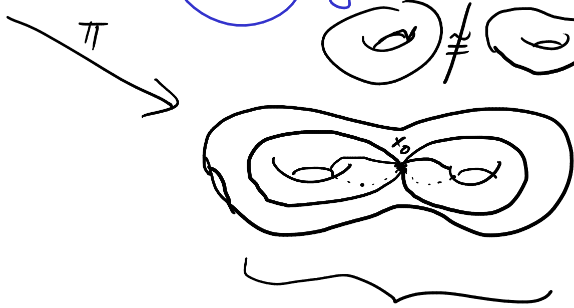


$\Rightarrow \pi_1 = \langle a_1,b_1,\dots,a_g,b_g \mid \prod_{i=1}^{g} a_i b_i a_i^{-1} b_i^{-1} = 1\rangle$

For different g, these groups are not isomorphic!







g "holes"

$\updownarrow$ homeomorphic