Solutions 1

Exercise 1. Let X, Y be sets, $A, A' \subset X$ and $B, B' \subset Y$, and $f\colon X \to Y$ a map. Show that

a) $A \subset f^{-1}(f(A))$ and equality holds if f is injective,

b) $f(f^{-1}(B)) \subset B$ and equality holds if f is surjective,

c) $f^{-1}(B \cup B') = f^{-1}(B) \cup f^{-1}(B')$,

d) $f^{-1}(B \cap B') = f^{-1}(B) \cap f^{-1}(B')$,

e) $f(A \cup A') = f(A) \cup f(A')$,

f) $f(A \cap A') \subset f(A) \cap f(A')$.

Find an example where $f(A \cap A') \neq f(A) \cap f(A')$.

Solution 1.

a) If $x \in A$, then $f(x) \in f(A)$, so $x \in f^{-1}(f(A))$. If $x \in f^{-1}(f(A))$ and f is injective, then $f(x) \in f(A)$, so $f(x) = f(y)$ for some $y \in A$, and $x = y$ by injectivity, so $x \in A$.

b) If $x \in f(f^{-1}(B))$, then $x = f(y)$ for some $y \in f^{-1}(B)$, that is some $y \in A$ with $f(y) \in B$. So $x \in B$. Conversely, if $x \in B$ and f is surjective, then by surjectivity $x = f(y)$ for some $y \in A$, and $y \in f^{-1}(B)$ since $f(y) \in B$. So $x \in f(f^{-1}(B))$.

c) $x \in f^{-1}(B \cup B')$ means that $f(x) \in B \cup B'$ and $x \in f^{-1}(B) \cup f^{-1}(B')$ means that either $f(x) \in B$ or $f(x) \in B'$, which is equivalent.

the others are similar. A possible example for $f(A \cap A') \neq f(A) \cap f(A')$ would be a constant function $f(x) = c$ and disjoint sets A and A'. Then $f(A \cap A') = \varnothing$ but $f(A) \cap f(A') = \{c\}$.

Exercise 2. Prove that the composition of injective/surjective/bijective maps is injective/surjective/bijective.

Solution 2. If $f\colon B \to C$ and $g\colon A \to B$ are injective and $f(g(x)) = f(g(y))$, then $g(x) = g(y)$ and therefore $x = y$. so $f \circ g$ is injective. If both are surjective and $z \in C$, then there exists $y \in B$ with $z = f(y)$ and there exists $x \in A$ with $y = g(x)$, so $z = f(g(x))$, i.e. $f \circ g$ is surjective. If f and g are bijective, they are both injective and surjective, so $f \circ g$ is injective and surjective, and therefore bijective.

Exercise 3 (Cantor–Bernstein–Schröder)**.** Let $f\colon A \to B$ and $g\colon B \to A$ be injective maps. We define recursively the sets

$$C_0 = A \setminus g(B), \qquad C_{n+1} = g(f(C_n)), \qquad C = \bigcup_{n=0}^{\infty} C_n$$

and a new map $h\colon A \to B$ by

$$h(x) = \begin{cases} f(x) & \text{if } x \in C, \\ g^{-1}(x) & \text{if } x \notin C, \end{cases}$$

where the preimage $g^{-1}(x)$ is well–defined since g is injective and $x \in g(B)$ in that case (check that!). Show that h is bijective.

Conclude that if $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.

Solution 3. h is well–defined since if $x \notin C$ then $x \notin C_0$, so $x \in g(B)$.

Also note that $g(f(C)) \cup C_0 = C$, and in particular $g(f(C)) \subset C$.

We want to show that h is injective. Assume $h(x) = h(y)$ for some $x, y \in A$. If x and y are either both in C or both not in C, then injectivity follows from that of f or the fact that g is a function. So assume exactly one of them is in C. Wlog $x \in C$ and $y \notin C$. Then $g(f(x)) = g(h(x)) = g(h(y)) = g(g^{-1}(y)) = y$. This is a contradiction since $g(f(C)) \subset C$.

Now we show that h is surjective. Let $y \in B$. If $g(y) \notin C$, $h(g(y)) = g^{-1}(g(y)) = y$. On the other hand, if $g(y) \in C$, then either $g(y) \in C_0$ or $g(y) \in g(f(C))$. The first option is impossible by the definition of C_0, so $g(y) = g(f(x))$ for some $x \in C$. Since g is injective, this implies $y = f(x) = h(x)$.

Exercise 4. To solve this, you might need some tools from Real Analysis.

a) Show that the map

$$f\colon \mathbb{R} \to \mathcal{P}(\mathbb{Q}), \quad x \mapsto \{q \in \mathbb{Q} \mid q \leq x\}$$

is injective.

b) Let $\{0,1\}^{\mathbb{N}}$ denote the set of infinite sequences $(a_1, a_2, \dots)$ with $a_i \in \{0,1\}$ for all i. Show that the map

$$g\colon \{0,1\}^{\mathbb{N}} \to \mathbb{R}, \quad (a_i)_{i\in\mathbb{N}} \mapsto 2\sum_{i=1}^{\infty} 3^{-i} a_i$$

is injective. Its image $C = g(\{0,1\}^{\mathbb{N}})$ is called the *Cantor set*. We will meet it again later in the course.

c) Show that $|\{0,1\}^{\mathbb{N}}| = |\mathcal{P}(\mathbb{N})| = |\mathcal{P}(\mathbb{Q})|$.

d) Use the previous parts to show

$$|C| = |\mathbb{R}| = |\mathcal{P}(\mathbb{N})|.$$

Solution 4.

a) Assume that $f(x) = f(y)$ but $x < y$. It is known that there exists a rational number $q \in \mathbb{Q}$ with $x < q < y$. But then $q \in f(y)$ and $q \notin f(x)$, a contradiction.

b) Assume that $(a_i)_{i\in\mathbb{N}}, (b_i)_{i\in\mathbb{N}} \in \{0,1\}^{\mathbb{N}}$ are two sequences which are not equal. Let $k \in \mathbb{N}$ be the first index where they differ, i.e. $a_k \neq b_k$ but $a_i = b_i$ for all $i < k$. Assume that g maps both sequences to the same number. Then

$$2\sum_{i=1}^{\infty} 3^{-i}a_i = 2\sum_{i=1}^{\infty} 3^{-i}b_i \quad \Rightarrow \quad (a_k - b_k) + \sum_{i=k+1}^{\infty} 3^{k-i}(a_i - b_i) = 0$$

and therefore (since $|a_i - b_i| \leq 1$ for all i)

$$1 = |a_k - b_k| = \left|\sum_{i=k+1}^{\infty} 3^{k-i}(a_i - b_i)\right| \leq \sum_{i=k+1}^{\infty} 3^{k-i} = \sum_{i=1}^{\infty} 3^{-i} = \frac{1}{2}$$

which is a contradiction. (the series is a geometric series. If you don't know how to compute these, look it up! It is very useful.)

c) We know that a bijection $\mathbb{N} \to \mathbb{Q}$ exists, and this induces a bijection $\mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{Q})$. Define a map $\mathcal{P}(\mathbb{N}) \to \{0,1\}^{\mathbb{N}}$ by sending a set $A \subset \mathbb{N}$ to the sequence $(a_i) \in \{0,1\}^{\mathbb{N}}$ which is defined by $a_i = 1$ if $i \in A$ and $a_i = 0$ if $i \notin A$. It is easy to show that this map is bijective.

d) a) showed that $|\mathbb{R}| \leq |\mathcal{P}(\mathbb{Q})|$ and b) showed that $|\{0,1\}^{\mathbb{N}}| = |C| \leq |\mathbb{R}|$. Together with c) we get

$$|\mathbb{R}| \leq |\mathcal{P}(\mathbb{Q})| = |\mathcal{P}(\mathbb{N})| = |\{0,1\}^{\mathbb{N}}| = |C| \leq |\mathbb{R}|,$$

so all of these cardinalities are equal.