Solutions 2

Exercise 1. Let X be a topological space with basis $\mathcal{B}$ and $Y \subset X$. Show that

$$\mathcal{B}_Y = \{B \cap Y \mid B \in \mathcal{B}\}$$

is a basis for the subspace topology on Y.

Solution 1. Let $\mathcal{T}$ be the topology on X and $\mathcal{T}_Y$ the subspace topology on Y. Let $U \in \mathcal{T}_Y$ and $x \in U$. By the definition of $\mathcal{T}_Y$ there is $V \in \mathcal{T}$ with $U = V \cap Y$. Since $x \in V \in \mathcal{T}$ there is a $W \in \mathcal{B}$ with $x \in W \subset V$. But then also $x \in W \cap Y \subset V \cap Y = U$, and $W \cap Y \in \mathcal{B}_Y$. So $\mathcal{B}_Y$ is a basis for $\mathcal{T}_Y$.

Exercise 2. Let

$$\mathcal{B}_B = \{B_\varepsilon(x) \mid \varepsilon \in \mathbb{R}^+, x \in \mathbb{R}^n\}, \qquad B_\varepsilon(x) = \{y \in \mathbb{R}^n \mid \sum_{i=1}^n (y_i - x_i)^2 < \varepsilon\}$$

be the set of open balls in $\mathbb{R}^n$ and let

$$\mathcal{B}_R = \{R(u,v) \mid u, v \in \mathbb{R}^n\}, \qquad R(u,v) = \{x \in \mathbb{R}^n \mid \forall i \leq n \colon u_i < x_i < v_i\}$$

be the set of open boxes (or rectangles) in $\mathbb{R}^n$.

a) Show that $\mathcal{B}_B$ and $\mathcal{B}_R$ are each a basis of a topology on $\mathbb{R}^n$.

b) Show that the topologies generated by $\mathcal{B}_B$ and $\mathcal{B}_R$ are equal.

Solution 2. We already discussed a) in class, so let's focus on b). We want to show that the topology generated by $\mathcal{B}_B$ is finer and also coarser than that generated by $\mathcal{B}_R$. Let $B_\varepsilon(x) \in \mathcal{B}_B$ and $y \in B_\varepsilon(x)$. Write $\delta = \varepsilon - \|x - y\|$. Then $B_\delta(y) \subset B_\varepsilon(x)$. Define

$$R = (y_1 - n^{-1/2}\delta, y_1 + n^{-1/2}\delta) \times \cdots \times (y_n - n^{-1/2}\delta, y_n + n^{-1/2}\delta).$$

Then $R \in \mathcal{B}_R$, $y \in R$ and also $R \subset B_\delta(y) \subset B_\varepsilon(x)$, because for every $z \in R$

$$\|y - z\|^2 = \sum_{i=1}^n |y_i - z_i|^2 < \sum_{i=1}^n (n^{-1/2}\delta)^2 = \delta^2.$$

Now let $R(u, v) \in \mathcal{B}_R$ and $x \in R(u, v)$. Let

$$\delta = \min \{x_i - u_i \mid i \leq n\} \cup \{v_i - x_i \mid i \leq n\}.$$

Then $\delta > 0$, $x \in B_\delta(x)$, $B_\delta(x) \in \mathcal{B}_B$ and $B_\delta(x) \subset R(u, v)$ since, for every $y \in B_\delta(x)$ and every i,

$$y_i - u_i = (x_i - u_i) - (x_i - y_i) > \delta - \delta = 0$$

and

$$v_i - y_i = (v_i - x_i) - (y_i - x_i) > \delta - \delta = 0,$$

so $u_i < y_i < v_i$, i.e. $y \in R(u, v)$.

Exercise 3. Show that the countable set

$$\{(a, b) \times (c, d) \mid a, b, c, d \in \mathbb{Q}, a < b, c < d\}$$

is a basis for the standard topology on $\mathbb{R}^2$.

Solution 3. Call this set $\mathcal{B}'$ and $\mathcal{B}$ the set of all open boxes. Let $\mathcal{T}$ be the (standard) topology generated by $\mathcal{B}$ and $\mathcal{T}'$ the topology generated by $\mathcal{B}'$. We want to show $\mathcal{T} = \mathcal{T}'$. Clearly $\mathcal{T}$ is finer than $\mathcal{T}'$ since $\mathcal{B}' \subset \mathcal{B}$. So we only need to show that $\mathcal{T}'$ is finer than $\mathcal{T}$. Let $U \in \mathcal{B}$ and $x \in U$. We can write $U = (a, b) \times (c, d)$ with $a, b, c, d \in \mathbb{R}$. So $a < x_1 < b$ and $c < x_2 < d$. There are rational numbers $a', b', c', d' \in \mathbb{Q}$ with $a < a' < x_1 < b' < b$ and $c < c' < x_2 < d' < d$. So

$$x \in (a', b') \times (c', d') \subset (a, b) \times (c, d),$$

and $(a', b') \times (c', d') \in \mathcal{B}'$. By Lemma 1.3, this implies that $\mathcal{T}'$ is finer than $\mathcal{T}$, so $\mathcal{T} = \mathcal{T}'$.

Exercise 4. Consider the following topologies on $\mathbb{R}$:

$\mathcal{T}_1$ = the standard topology
$\mathcal{T}_2$ = the finite complement topology
$\mathcal{T}_3$ = the lower limit topology, having all sets $[a, b)$ as basis
$\mathcal{T}_4$ = the upper limit topology, having all sets $(a, b]$ as basis
$\mathcal{T}_5$ = the topology having all sets $(-\infty, a)$ as basis

Determine which are these topologies are finer / coarser than which of the others.

Definition of the *finite complement topology*: the open sets are all sets whose complement in X is finite (and the empty set).

Solution 4. $\mathcal{T}_2$ and $\mathcal{T}_5$ are (strictly) coarser than the standard topology, but not comparable to each other, and $\mathcal{T}_3$ and $\mathcal{T}_4$ are (strictly) finer than the standard topology, but not comparable to each other. Proof:

- First of all, none of the topologies $\mathcal{T}_2, \mathcal{T}_3, \mathcal{T}_4, \mathcal{T}_5$ are equal to the standard topology, since $[0,1)$ and $(0,1]$ are open in $\mathcal{T}_3$ and $\mathcal{T}_4$ respectively, but not in $\mathcal{T}_1$, and $(0,1)$ is open in $\mathcal{T}_1$ but neither in $\mathcal{T}_2$ nor $\mathcal{T}_5$.
- Let $F \subset \mathbb{R}$ be finite and $\mathbb{R} \setminus F$ the corresponding open set in $\mathcal{T}_2$. Let $x \in \mathbb{R} \setminus F$. Pick a to be the maximal element of F which is less than x (or $x-1$ if no element of F is less than x) and pick b to be the minimal element of F which is greater than x (or $x+1$ if no element of F is greater than x). Then $x \in (a,b) \subset \mathbb{R} \setminus F$. So $\mathcal{T}_1$ is finer than $\mathcal{T}_2$.
- Let $x \in (a,b)$. Then $x \in [x,b) \subset (a,b)$, so $\mathcal{T}_3$ is finer than $\mathcal{T}_1$.
- Let $x \in (a,b)$. Then $x \in (a,x] \subset (a,b)$, so $\mathcal{T}_4$ is finer than $\mathcal{T}_1$.
- Let $x \in (-\infty,a)$. Then $x \in (x-1,a) \subset (-\infty,a)$, so $\mathcal{T}_1$ is finer than $\mathcal{T}_5$.
- The set $(-\infty,0)$ is open in $\mathcal{T}_5$, but not in $\mathcal{T}_2$. The set $\mathbb{R} \setminus \{0\}$ is open in $\mathcal{T}_2$, but not in $\mathcal{T}_5$ (the open sets in $\mathcal{T}_5$ are just $\varnothing$, $\mathbb{R}$ and $(-\infty,a)$ for $a \in \mathbb{R}$). So these topologies are uncomparable.
- The set $[0,1)$ is open in $\mathcal{T}_3$, but not in $\mathcal{T}_4$: if it was open in $\mathcal{T}_4$ there would be an interval of the form $(a,b]$ such that $0 \in (a,b] \subset [0,1)$. This is impossible, as it would imply $a < 0$ and $a \geq 0$. By an analogous argument, $(0,1]$ is open in $\mathcal{T}_4$, but not in $\mathcal{T}_3$.