Solutions 3

Exercise 1. Let X, Y be topological spaces and $A \subset X$, $B \subset Y$ subsets. Show that if you take the product topology on $X \times Y$ and then the subspace topology on $A \times B \subset X \times Y$ you obtain the same topology as if you first take the subspace topologies on A and B and then the product topology on $A \times B$.

Solution 1. Write $\mathcal{T}$ for the subspace topology on $A \times B$ induced by the product topology on $X \times Y$, and write $\mathcal{T}'$ for the product topology on $A \times B$ induced by the subspace topologies on X and Y. Let $\mathcal{T}_X$ be the topology on X and $\mathcal{T}_Y$ the topology on Y.

Then $\mathcal{B}_{X\times Y} = \{U \times V \mid U \in \mathcal{T}_X, V \in \mathcal{T}_Y\}$ is a basis of the product topology on $X \times Y$. By Exercise 1 on Sheet 2, this means that

$$\mathcal{B} = \{W \cap (A \times B) \mid W \in \mathcal{B}_{X\times Y}\} = \{(U \times V) \cap (A \times B) \mid U \in \mathcal{T}_X, V \in \mathcal{T}_Y\}$$

is a basis of $\mathcal{T}$.

On the other hand, the subspace topologies $\mathcal{T}_A$ and $\mathcal{T}_B$ on A and B are given by

$$\mathcal{T}_A = \{U \cap A \mid U \in \mathcal{T}_X\}, \quad \mathcal{T}_B = \{V \cap B \mid V \in \mathcal{T}_Y\},$$

so a basis of $\mathcal{T}'$ is

$$\mathcal{B}' = \{U' \times V' \mid U' \in \mathcal{T}_A, V' \in \mathcal{T}_B\} = \{(U \cap A) \times (V \cap B) \mid U \in \mathcal{T}_X, V \in \mathcal{T}_Y\}.$$

But $(U \times V) \cap (A \times B) = (U \cap A) \times (V \cap B)$, so $\mathcal{B} = \mathcal{B}'$ and therefore $\mathcal{T} = \mathcal{T}'$.

Exercise 2. Let X be a topological space and $A, B \subset X$. Show that

a) if $A \subset B$, then $\overline{A} \subset \overline{B}$.

b) $\overline{A \cup B} = \overline{A} \cup \overline{B}$

c) $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$

d) Find a counterexample for $\overline{A \cap B} = \overline{A} \cap \overline{B}$.

Solution 2.

a) Let $\mathcal{C}_A, \mathcal{C}_B \subset \mathcal{P}(X)$ be the set of closed subsets containing A and B, respectively. So $\overline{A} = \bigcap \mathcal{C}_A$ and $\overline{B} = \bigcap \mathcal{C}_B$. Since $A \subset B$ every closed set containing B also contains A, i.e. $\mathcal{C}_B \subset \mathcal{C}_A$. But this means $\bigcap \mathcal{C}_A \subset \bigcap \mathcal{C}_B$ since intersecting over fewer sets gives a bigger set.

b) $A \subset A \cup B$ and $B \subset A \cup B$, so $\overline{A} \subset \overline{A \cup B}$ and $\overline{B} \subset \overline{A \cup B}$ and thus $\overline{A} \cup \overline{B} \subset \overline{A \cup B}$. On the other hand, $A \subset \overline{A}$ and $B \subset \overline{B}$, so $A \cup B \subset \overline{A} \cup \overline{B}$. Since $\overline{A} \cup \overline{B}$ is closed, this implies $\overline{A \cup B} \subset \overline{A} \cup \overline{B}$.

c) If $x \in A \cap B$, then $x \in A \subset \overline{A}$ and $x \in B \subset \overline{B}$, so $x \in \overline{A} \cap \overline{B}$. so $A \cap B \subset \overline{A} \cap \overline{B}$, and since $\overline{A} \cap \overline{B}$ is closed, this implies $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$.

d) $A = \mathbb{Q}$ and $B = \mathbb{R} \setminus \mathbb{Q}$ are both dense in $\mathbb{R}$, so $\overline{A} \cap \overline{B} = \mathbb{R} \cap \mathbb{R} = \mathbb{R}$, but $A \cap B = \varnothing$, so $\overline{A \cap B} = \varnothing$.

Exercise 3. Let X be a topological space. Show that X is Hausdorff if and only if the diagonal

$$\Delta = \{(x, x) \mid x \in X\}$$

is closed in the product topology on $X \times X$.

Solution 3. First assume that Δ is closed in the product topology and let $x, y \in X$ be distinct. We want to find disjoint open neighborhoods of them. Note that $(x, y) \in X \times X \setminus \Delta$. Since this set is open, there is a set of the form $U \times V$ with $U, V \subset X$ open such that $(x, y) \in U \times V \subset X \times X \setminus \Delta$. U and V are open neighborhoods of x and y, and they are disjoint, as any intersection would yield a point in $(U \times V) \cap \Delta$. This proves that X is Hausdorff.

Now assume X is Hausdorff and let $(x, y) \in X \times X \setminus \Delta$. This means $x \neq y$, so there are disjoint open neighborhoods U of x and V of y. The disjointness can be reformulated as $(U \times V) \cap \Delta = \varnothing$. So $(x, y) \in U \times V \subset X \times X \setminus \Delta$, and $U \times V$ is open in the product topology. Since we can do this for every $(x, y) \in X \times X \setminus \Delta$, we see that $X \times X \setminus \Delta$ is open.

Exercise 4. Recall the topologies on $\mathbb{R}$ from the last homework:

$\mathcal{T}_2 =$ the finite complement topology

$\mathcal{T}_3 =$ the lower limit topology, having all sets $[a, b)$ as basis

$\mathcal{T}_4 =$ the upper limit topology, having all sets $(a, b]$ as basis

$\mathcal{T}_5 =$ the topology having all sets $(-\infty, a)$ as basis

Describe for each of them which sequences converge to which limits.

Solution 4. In $\mathcal{T}_2$, if a sequence x_n contains exactly one number $x \in \mathbb{R}$ infinitely many times, then it converges to x. If it contains more than one element infinitely many times, it does not converge. If it contains no element infinitely many times, it converges to every element of $\mathbb{R}$.

In $\mathcal{T}_3$, a sequence x_n converges to x if and only if it converges to x in the standard topology and almost every element of the sequence is greater or equal than x.

In $\mathcal{T}_4$, a sequence x_n converges to x if and only if it converges to x in the standard topology and almost every element of the sequence is less or equal than x.

In $\mathcal{T}_5$, a sequence x_n converges to x if and only if it is bounded above and no subsequence converges to an element greater than x in the standard topology. In other words: if x_n is not bounded above, it does not converge. If it is bounded above, it converges to every number greater or equal than its greatest accumulation point (limit of a subsequence in the standard topology).

Exercise 5. Show that the intersection of dense sets need not be dense, but the intersection of two open dense sets is always open and dense.

Solution 5. Both $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are dense in $\mathbb{R}$, but their intersection is empty.

Let $U, V \subset X$ be open dense subsets of a topological space X. Then $W = X \setminus \overline{U \cap V}$ is open, and $W \cap U \cap V = \varnothing$. If $W \cap U$ was non–empty, it would have to intersect the dense set V, so $W \cap U = \varnothing$. Similarly, if W was non–empty, it would have to intersect the dense set U, so $W = \varnothing$. This means $U \cap V$ is dense.