Solutions 4

Exercise 1. Let X, Y be topological spaces, $f, g\colon X \to Y$ functions from X to Y, $U \subset X$ an open set, and $x \in U$. Assume that $f|_U = g|_U$. Show that f is continuous at x if and only if g is continuous at x.

This shows that whether a function is continuous at x only depends on its behavior in an (arbitrarily small) open neighborhood of x. We say that continuity at a point is a *local property.*

Solution 1. Assume that f is continuous at x. We want to show that g is continuous at x, so let V be a neighborhood of $g(x) = f(x)$. There is a neighborhood W of x such that $f(W) \subset V$. Let $Z = U \cap W$. Then Z is alos a neighborhood of x, and $g(Z) = f(Z) \subset f(W) \subset V$. So g is continuous at x.

Exercise 2. Let X, Y be topological spaces and $A \subset X$.

a) Show that the subspace topology on A is the coarsest topology such that the inclusion map $\iota\colon A \to X$ is continuous.

b) Show that the product topology on $X \times Y$ is the coarsest topology such that the projections $\pi_1\colon X \times Y \to X$ and $\pi_2\colon X \times Y \to Y$ are continuous.

Solution 2. Let X be a set and $f_i\colon X \to Y_i$ for $i \in I$ a family of maps to topological spaces Y_i indexed by some set I. Let $\mathcal{T}$ be the topology on X generated by all sets of the form $f_i^{-1}(U)$ for $U \subset Y_i$ open and $i \in I$. Then the f_i are clearly continuous in this topology, and every other topology $\mathcal{T}'$ on X which makes these maps continuous must also contain the sets $f_i^{-1}(U)$, so $\mathcal{T} \subset \mathcal{T}'$. So $\mathcal{T}$ is the coarsest topology making the f_i continuous.

a) According to the preceding paragraph, we have to show that the subspace topology is generated by the sets $\iota^{-1}(U)$ for all $U \subset X$ open. But $\iota^{-1}(U) = U \cap A$, so these sets already make up the whole topology.

b) Again we have to show that the product topology is generated by the sets $\pi_1^{-1}(U_1)$ and $\pi_2^{-1}(U_2)$ for $U_1 \subset X$ and $U_2 \subset Y$. We already showed this in class: by definition of the product topology, the intersections $\pi_1^{-1}(U_1) \cap \pi_2^{-1}(U_2)$ form a basis.

Exercise 3. Let $f\colon X \to Y$ be a continuous map between topological spaces X and Y. Show that if a sequence $x_n \in X$ converges to x, then $f(x_n)$ converges to $f(x)$.

Solution 3. Let $x_n \to x$ be a convergent sequence in X. Since f is continuous, it is continuous at x. Let V be a neighborhood of $f(x)$. Then there is a neighborhood U of x such that $f(U) \subset V$. Since $x_n \in U$ for almost every n, we get $f(x_n) \in f(U) \subset V$ for almost every n. This is true for every neighborhood V, so $f(x_n) \to f(x)$.

Exercise 4. Let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of connected subspaces of X, with the property that $A_n \cap A_{n+1} \neq \varnothing$ for every $n \in \mathbb{N}$. Show that $\bigcup_{n\in\mathbb{N}} A_n$ is connected.

Solution 4. Assume that $\bigcup_{n\in\mathbb{N}} A_n$ is not connected, so it contains a non–empty clopen subset C with non–empty complement. If $A_n \cap C \neq \varnothing$ for any $n \in \mathbb{N}$, then $A_n \cap C$ is clopen in A_n and non–empty, so $A_n \cap C = A_n$, or equivalently $A_n \subset C$. So every A_n is either disjoint from C or contained in C. Since the A_n can not all by disjoint from C or all contained in C, there is some $n \in \mathbb{N}$ such that $A_n \cap C = \varnothing$ but that A_{n-1} or A_{n+1} is contained in C. This contradicts $A_n \cap A_{n+1} \neq \varnothing$.

Exercise 5. A space X is called *totally disconnected* if the only connected subspaces are the one–point sets.

a) If X has the discrete topology, show that it is totally disconnected.

b) Show that $\mathbb{Q} \subset \mathbb{R}$ is totally disconnected.

c) Show that the Cantor set $C \subset \mathbb{R}$ is totally disconnected (see exercise sheet 1 for the definition).

Hint for c): To get an intuition for the Cantor set, convince yourself that it can be constructed iteratively in the following way: take the unit interval $[0,1]$ *and remove the (open) middle third. The remainder is the set* $[0,\frac{1}{3}] \cup [\frac{2}{3},1]$*. Now perform the same operation on each of its parts and obtain* $[0,\frac{1}{9}] \cup [\frac{2}{9},\frac{1}{3}] \cup [\frac{2}{3},\frac{7}{9}] \cup [\frac{8}{9},1]$*. If we continue this procedure infinitely and then take the intersection over all steps, we get* C*. Here is a picture of the first 7 steps:*

Solution 5.

a) Every subset of X which has two or more elements has non–trivial clopen subsets (just take one of its elements and its complements) and is therefore not connected.

b) Let $A \subset \mathbb{Q}$ be a set which contains at least two different elements $x, y \in A$. Assume that $x < y$. Then there exists an irrational number $z \in \mathbb{R} \setminus \mathbb{Q}$ with $x < z < y$. The sets $(-\infty, z) \cap A$ and $(z, \infty) \cap A$ are disjoint, both open in A and their union is A. Also both are non–empty, since $x \in (-\infty, z) \cap A$ and $y \in (z, \infty) \cap A$. So A is not connected.

c) We want to use the same argument as for $\mathbb{Q}$. To do this, we just have to find, for every pair of distinct points $x, y \in C$ with $x < y$, a $z \in \mathbb{R} \setminus C$ which is is in–between, i.e. $x < z < y$.

   Recall that the elements of the Cantor set are parametrized by sequences $(a_i) \in \{0,1\}^{\mathbb{N}}$. Such a sequence corresponds to the number $x = 2\sum_{i=1}^{\infty} a_i 3^{-i} \in C$. Knowing the first k elements $a_1, \dots, a_k$ of the sequence gives a rough estimate on where the point x will be, namely in the interval

$$S = \left[2\sum_{i=1}^{k} a_i 3^{-i},\ 2\sum_{i=1}^{k} a_i 3^{-i} + 3^{-k} \right].$$

   This is because $2\sum_{i=k+1} 3^{-i} = 3^{-k}$. Note that if we change any of the $a_1, \dots, a_k$ the resulting number $2\sum_{i=1}^{k} a_i 3^{-i}$ will change by at least $2 \cdot 3^{-k}$, so the corresponding intervals S will be disjoint. Therefore, there are no elements of C in S except the ones whose sequence starts with $a_1, \dots, a_k$.

   Now let $x, y \in C$ with $x < y$. Suppose x is represented by the sequence $(a_i) \in \{0,1\}^{\mathbb{N}}$ and y be the sequence $(b_i) \in \{0,1\}^{\mathbb{N}}$. Then there is a first element $k \in \mathbb{N}$ where these sequences disagree, i.e. $a_k \neq b_k$ but $a_i = b_i$ for all $i < k$. This means that $a_k = 0$ and $b_k = 1$. Let $x_0 = 2\sum_{i=1}^{k-1} a_i 3^{-i}$. By the above, all sequences starting with $a_1, \dots, a_{k-1}$ correspond to numbers in $[x_0, x_0 + 3 \cdot 3^{-k}]$, and there are no other elements of C in this interval. If the next (k–th) element of the sequence is a 0, we end up in $[x_0, x_0 + 3^{-k}]$ (in particular, x is in this interval), and if the next element is a 1, we're in the open interval $[x_0 + 2 \cdot 3^{-k}, x_0 + 3 \cdot 3^{-k}]$ (so y is in this interval). But between them is the interval $(x_0 + 3^{-k}, x_0 + 2 \cdot 3^{-k})$ which can not contain any elements of C. So we can choose any z from this interval.