Solutions 5

Exercise 1. Let X be a topological space and Y a compact Hausdorff space. Show that a function $f\colon X \to Y$ is continuous if and only if its graph

$$\mathrm{gr}(f) = \{(x, f(x)) \mid x \in X\}$$

is a closed subset of $X \times Y$.

Hint: Show that the projection $X \times Y \to X$ maps closed sets to closed sets.

Solution 1. We first prove that the projection $\pi_1\colon X \times Y \to X$ is a closed map, i.e. it takes closed sets to closed sets. Let $K \subset X \times Y$ be closed and $x \in X \setminus \pi_1(K)$. Then $\{x\} \times Y$ is contained in the open set $N = (X \times Y) \setminus K$, and so by the tube lemma, there is a neighborhood W of x such that $W \times Y \subset N$, and therefore $W \subset X \setminus \pi_1(K)$. This shows that $X \setminus \pi_1(K)$ is open, so π_1 maps closed sets to closed sets.

To show that $\mathrm{gr}(f)$ being closed implies that f is continuous, let $C \subset Y$ be closed. It is easily checked that

$$f^{-1}(C) = \pi_1(\pi_2^{-1}(C) \cap \mathrm{gr}(f)).$$

Since C and $\mathrm{gr}(f)$ are closed, and π_2 is continuous, $\pi_2^{-1}(C) \cap \mathrm{gr}(f)$ is closed. By the above, π_1 maps this to a closed sets, so $f^{-1}(C)$ is closed. This shows that f is continuous.

Conversely, suppose that f is continuous. We show that $\mathrm{gr}(f)$ is closed by showing that its complement in $X \times Y$ is open. Let $(x, y) \in (X \times Y) \setminus \mathrm{gr}(f)$. Then $y \neq f(x)$, and becaue Y is Hausdorff there are disjoint neighborhoods U of y and V of $f(x)$. The inverse image $f^{-1}(V)$ is a neighborhood of x. We claim that $f^{-1}(V) \times U$ is a neighborhood of (x, y) which does not intersection $\mathrm{gr}(f)$, and therefore shows that $(X \times Y) \setminus \mathrm{gr}(f)$ is open. Indeed, if it intersected $\mathrm{gr}(f)$, there would be a $x' \in f^{-1}(V)$ with $f(x') \in U$, which is impossible since $f(x')$ would be both in V and U, but we assumed them to be disjoint.

Exercise 2. Consider $\mathbb{R}$ with the finite complement topology. Show that every subset is compact.

Solution 2. The subspace topology on any subset $A \subset \mathbb{R}$ is just the finite complement topology on A. So instead we show that any space X which carries the finite complement topology is compact. Let $\mathcal{C}$ be an open cover of X. Assuming X to be non–empty, take any $U \subset \mathcal{C}$. The set U is open, i.e. $U = X \setminus F$ for some finite set F. Now since every finite set is compact, there is a finite subset $\mathcal{C}'$ of $\mathcal{C}$ covering F. Then $\mathcal{C}' \cup \{U\}$ is an open subcover of $\mathcal{C}$.

Exercise 3. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two topologies on a set X which are both compact and Hausdorff. Show that either $\mathcal{T}_1 = \mathcal{T}_2$ or the two topologies are incomparable.

Solution 3. Assume that $\mathcal{T}_1 \subset \mathcal{T}_2$. Then the identity map $\mathrm{id}\colon (X, \mathcal{T}_2) \to (X, \mathcal{T}_1)$ is continuous. Let C be closed in $\mathcal{T}_2$ (i.e. $X \setminus C \in \mathcal{T}_2$). Since $\mathcal{T}_2$ is compact, C is compact as a subspace of $\mathcal{T}_2$. Because continuous maps map compact sets to compact sets, C is also compact as a subspace of $\mathcal{T}_1$. But $\mathcal{T}_1$ is Hausdorff, so this means that C is closed in $\mathcal{T}_1$. We showed that every closed set in $\mathcal{T}_2$ is closed in $\mathcal{T}_1$, so $\mathcal{T}_2 \subset \mathcal{T}_1$ and therefore $\mathcal{T}_1 = \mathcal{T}_2$. If we instead assume $\mathcal{T}_2 \subset \mathcal{T}_1$, the same argument also shows $\mathcal{T}_1 = \mathcal{T}_2$.

Exercise 4. Let X be compact, Y a Hausdorff space and $f\colon X \to Y$ continuous.

a) Show that f is a *closed map*, i.e. the image of every closed set is closed.

b) Show that if f is bijective, it is a homeomorphism.

Solution 4.

a) Let $C \subset X$ be closed. Then C is compact, hence $f(C)$ is compact. Since Y is Hausdorff space, $f(C)$ is closed.

b) Because f is closed, its inverse map is continuous (as every preimage of a closed set is closed). So f is continous, bijective and has a continuous inverse, i.e. f is a homeomorphism.

Exercise 5. Let X be a compact space and let $(C_i)_{i\in\mathbb{N}}$ be a sequence of non–empty closed subsets such that $C_{i+1} \subset C_i$ for every i. Show that the intersection $\bigcap_{i\in\mathbb{N}} C_i$ is non–empty.

Hint: Assume the contrary and look at the complements.

Solution 5. Assume that $\bigcap_{i\in\mathbb{N}} C_i = \varnothing$ and let $D_i = X \setminus C_i$ for every $i \in \mathbb{N}$. Then $D_i \subset D_{i+1}$ for every i, and $\bigcup_{i\in\mathbb{N}} D_i = X$, so D_i is an open cover of X. Since X is compact, there is a finite subcover, and because $D_i \subset D_{i+1}$ for all i, $D_n = X$ for some n. But this contradicts C_n being non–empty.

Definition 1. A *compactification* of a topological space X is a pair (Y, f) consisting of a topological space Y and a continuous map $f\colon X \to Y$ such that

- Y is compact,
- f is a homeomorphism onto $f(X)$,
- and $f(X)$ is dense in Y, i.e. $\overline{f(X)} = Y$.

Definition 2. A space X is *locally compact* if every $x \in X$ has an open neighborhood U and a compact set $K \subset X$ such that $U \subset K$.

Exercise 6.

a) Let $X = \mathbb{R}$ with the standard topology and let $Y = \mathbb{R} \cup \{-\infty, \infty\}$ equipped with the topology generated by all open intervals (a, b) as well as the sets $[-\infty, a)$ and $(a, \infty]$ for any $a \in \mathbb{R}$. Let $f\colon \mathbb{R} \to Y$ be the identity map. Show that (Y, f) is a compactification of $\mathbb{R}$.

b) Let X be a locally compact, but non–compact Hausdorff space. Let $Y = X \cup \{\infty\}$, and equip Y with a topology as follows: a subset $A \subset Y$ is open if

- either $\infty \notin A$ and A is open as a subset of X,
- or $\infty \in A$ and $Y \setminus A$ is a compact subset of X.

Let $f\colon X \to Y$ be the identity map. Show that (Y, f) is a compactification of X. It is called the *one–point compactification* of X.

c) Show that the one–point compactification of $\mathbb{R}$ is homeomorphic to S^1.

Solution 6.

a) While we can just check all the properties of a compactification manually, a maybe easier alternative is the following: Consider the map $f\colon [-1, 1] \to Y$ which maps ± 1 to $\pm\infty$ and $f(x) = x/(1 - x^2)$ for all $x \in (-1, 1)$. $[-1, 1]$ is clearly a compactification of $(-1, 1)$, so if we can show that f is a homeomorphism, we are done.

We know that f is bijective. To see that f and f^{-1} are continuous, we only have to check that the preimage and image of every element of some basis or subbasis is open. But we can easily see that

$$f((a, b)) = \left(\frac{a}{1 - a^2}, \frac{b}{1 - b^2}\right), \quad f((a, 1]) = \left(\frac{a}{1 - a^2}, \infty\right], \quad f([-1, b)) = \left[-\infty, \frac{b}{1 - b^2}\right).$$

These sets generate the topologies on $[-1, 1]$ and $\mathbb{R} \cup \{\pm\infty\}$, so f and f^{-1} are continuous.

b) Let us first prove that Y is compact. Let $\mathcal{A}$ be an open cover of Y. Pick $A \in \mathcal{A}$ with $\infty \in A$. Since A is open in Y, $Y \setminus A$ is a compact subset of X. Since $\mathcal{A}$ covers Y, $\mathcal{A} \setminus \{A\}$ covers $Y \setminus A$. By compactness, it has a finite subcover $\mathcal{A}' \subset \mathcal{A} \setminus \{A\}$ of $Y \setminus A$. Then $\mathcal{A}' \cup \{A\}$ is still finite and covers Y.

Now we show that $f|_X$ is a homeomorphism to its image. That is equivalent to proving that the topology on X coincides with the topology on X as a subspace of Y. A set is open in this subspace topology if and only if it is either open in X or its complement is compact. But the latter case also implies that it is open in X because X is Hausdorff, so compact sets are closed.

Finally, we show that $\overline{X} = Y$. Because $Y = X \cup \{\infty\}$, we just have to show that X is not closed as a subset of Y, or that $\{\infty\}$ is not open. This is true because $Y \setminus \{\infty\} = X$ is not compact.

Note that we did not use the local compactness. However, local compactness is usually required in the definition of the one–point compactification because it ensures that the compactification Y is still Hausdorff: To separate ∞ from some point $x \in X$ by disjoint open sets in Y, we need to find an open set U and a compact set K in X such that $x \in U \subset K$. This is exactly what local compactness gives us.

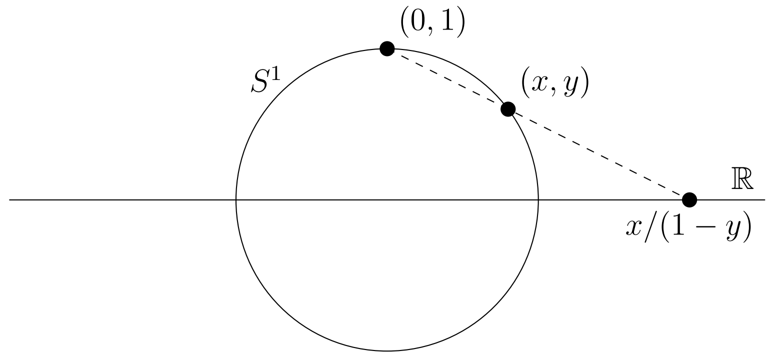
c) A possible homeomorphism is the map $f\colon S^1 \to \mathbb{R} \cup \{\infty\}$ which is given by $f(0,1) = \infty$ and $f(x,y) = x/(1-y)$ for all other $(x,y) \in S^1$. To see that f is bijective, we can compute its inverse map

$$f^{-1}\colon \mathbb{R} \cup \infty \to S^1, \quad t \mapsto \Big(\frac{2t}{t^2+1}, \frac{t^2-1}{t^2+1}\Big), \quad \infty \mapsto (0,1).$$

We want to show that f and f^{-1} are continuous. It is clear that $f|_{S^1 \setminus \{(0,1)\}}$ is a homeomorphism onto $\mathbb{R}$, so any open sets which do not contain ∞ resp. $(0,1)$ get mapped to open sets.

If $U \subset \mathbb{R} \cup \{\infty\}$ is open and contains ∞, then $\mathbb{R} \setminus U$ is compact in $\mathbb{R}$, so $f^{-1}(\mathbb{R} \setminus U)$ is compact in S^1, hence closed. So its complement $f^{-1}(U)$ is open in S^1. On the other hand, if $V \subset S^1$ is open and contains $(0,1)$, then $S^1 \setminus V$ is closed in S^1 and hence compact, and does not contain $(0,1)$. $f(S^1 \setminus V)$ is compact, meaning that its complement $f(V)$ is open by the definition of the topology on $\mathbb{R} \cup \{\infty\}$.

The choice of map f is explained by this picture. It is called *stereographic projection.*


(0, 1)
S^1
(x, y)
$\mathbb{R}$
x/(1 − y)