Solutions 6

Exercise 1. Let (X, d) be a metric space. For a subset $A \subset X$ and $x \in X$ define

$$\operatorname{dist}(x, A) = \inf_{y \in A} d(x, y).$$

a) Show that the map $X \to [0, \infty), x \mapsto \operatorname{dist}(x, A)$ is continuous.

b) Show that $\operatorname{dist}(x, A) = 0$ if and only if $x \in \overline{A}$.

c) Assume A is compact and let $x \in X$. Show that there exists $y \in A$ with

$$d(x, y) = \operatorname{dist}(x, A).$$

Give a counterexample to this when A is not compact.

Solution 1.

a) Let $x \in X$ and $\delta = \operatorname{dist}(x, A)$. We show that $\operatorname{dist}(-, A)$ is continuous at x. Let $V \subset \mathbb{R}$ be an open neighborhood of δ. We need to show that there is a neighborhood U of x whose image is contained in V. Because V is open, it contains an interval of the form $(\delta - \varepsilon, \delta + \varepsilon)$ for some $\varepsilon > 0$. For every $y \in B_\varepsilon(x)$ we have

$$\begin{aligned} \operatorname{dist}(y, A) &\leq d(x, y) + \operatorname{dist}(x, A) < \operatorname{dist}(x, A) + \varepsilon, \\ \operatorname{dist}(x, A) &\leq d(x, y) + \operatorname{dist}(y, A) < \operatorname{dist}(y, A) + \varepsilon. \end{aligned}$$

So $\operatorname{dist}(y, A) \in (\delta - \varepsilon, \delta + \varepsilon) \subset V$. Hence the map maps $B_\varepsilon(x)$ into V, so it is continuous at x.

b) We can use the description of the closure $\overline{A}$ as the set of limits of sequences in A. If $\operatorname{dist}(x, A) = 0$, there must be $y_n \in A$ with $d(x, y_n) < 1/n$ for every $n \in \mathbb{N}$. Clearly $y_n \to x$, so $x \in \overline{A}$. On the other hand, if $x \in \overline{A}$, there is a sequence $y_n \in A$ with $y_n \to x$ and hence $d(x, y_n) \to 0$. In computing $\operatorname{dist}(x, A)$ we take the infimum over a set which contains all these $d(x, y_n)$, so the infimum must be 0.

c) The map $y \mapsto d(x, y)$ is continuous (essentially by definition of the metric topology), so the compact set A maps to a compact subset of $\mathbb{R}$. Such a set has a minimum, i.e. an element $d(x, y)$ which achieves the infimum $\operatorname{dist}(x, A)$.

Exercise 2. Show that every subspace and every (finite) product of first/second countable spaces are first/secound countable.

Solution 2. If we have a basis of X and a subset $A \subset X$ we get a basis of A by intersecting each element of the basis with A. If the original basis was countable, the new one is countable too (it can't contain more sets).

To get a countable basis of the product $X \times Y$ take a countable basis $(B_i)_{i\in\mathbb{N}}$ of X and $(C_i)_{i\in\mathbb{N}}$ of Y and then set

$$\mathcal{B} = \{B_i \times C_j \mid i, j \in \mathbb{N}\}.$$

This set is countable since $\mathbb{N} \times \mathbb{N}$ is countable. It is easy to check that this is a basis of the product topology.

This shows that subspaces and finite products of second countable spaces are second countable. For first countability, apply the same arguments to neighborhood bases.

Exercise 3. Let (X, d) be a metric space. Show that

$$\overline{d}(x, y) = \begin{cases} d(x, y) & \text{if } d(x, y) < 1 \\ 1 & \text{otherwise} \end{cases}$$

defines another metric $\overline{d}$ on X, and that d and $\overline{d}$ induce the same topology.

Solution 3. Let's start by proving the following useful fact: If d, d' are two metrics on space X and for every $x \in X$ and every $\epsilon > 0$ there is a $\delta > 0$ such that $B'_\delta(x) \subset B_\epsilon(x)$ (here B' means the ball in the metric d' and B the ball in the metric d), then the topology induced by d' is finer (or equal) than that induced by d.

To show this fact, let $U \subset X$ be open in the topology coming from d, and $x \in U$. Then $B_\epsilon(x) \subset U$ for some $\epsilon > 0$ and therefore $x \in B'_\delta(x) \subset B_\epsilon(x) \subset U$ for some $\delta > 0$. Since this is true for any x, U is also open in the topology induced by d'.

Now we write $B^d_R(x)$ for an open ball in the metric d and $B^{\overline{d}}_R(x)$ for an open ball in the metric $\overline{d}$. Since $\overline{d}(x, y) \le d(x, y)$ for all $x, y \in X$ we have $B^d_\epsilon(x) \subset B^{\overline{d}}_\epsilon(x)$ for all $x \in X$. This shows that d generates a finer topology than $\overline{d}$. Conversely, if $\epsilon < 1$ then $B^{\overline{d}}_\epsilon(x) = B^d_\epsilon(x)$ and if $\epsilon \ge 1$ then $B^{\overline{d}}_{1/2}(x) = B^d_{1/2}(x) \subset B^d_\epsilon(x)$. So $\overline{d}$ also induces a finer topology than d, i.e. the topologies are equal.

Exercise 4. A topological space X is called *separable* if it contains a countable dense subset. Show that a metric space X is second countable if and only if it is separable.

Solution 4. If X is a second countable space, take a countable basis and choose one point in each element of that basis. The set A of these points is dense since every open set contains a basis element and therefore intersects A. The cardinality of A is at most that of the basis, so A is countable.

Conversely, given a metric space with a countable dense subset $D = \{x_1, x_2, \dots\} \subset X$, take the collection

$$\mathcal{B} = \{B_{1/n}(x_i) \mid i, n \in \mathbb{N}\}.$$

This is countable since $\mathbb{N} \times \mathbb{N}$ is countable, and it is a basis: Let $x \in U \subset X$ with U open. Then $B_\varepsilon(x) \subset U$ for some $\varepsilon > 0$. We can assume $\varepsilon < 3/2$. Because D is dense it intersects every open set, so $x_j \in B_{\varepsilon/3}(x)$ for some j. Choose $n \in \mathbb{N}$ such that $3/(2\varepsilon) < n < 3/\varepsilon$ (this is always possible if $\varepsilon < 3/2$). Then

$$x \in B_{\varepsilon/3}(x_j) \subset B_{1/n}(x_j) \subset B_{2\varepsilon/3}(x_j) \subset B_\varepsilon(x) \subset U.$$

So $\mathcal{B}$ is indeed a basis of X.

Exercise 5. An *isolated point* in a topological space X is a point $x \in X$ so that $\{x\}$ is open.

a) Show that X has the discrete topology if and only if every point of X is isolated.

b) Show that if X is compact and every point in X is isolated, then X is finite.

c) Show that a subspace of $\mathbb{R}^n$ can only have countably many isolated points.

Solution 5.

a) If X has the discrete topology, every set is open, so in particular the one element sets are open. Conversely, if all one element sets are open, then every set is open, because it can be written as a union of one element sets.

b) If X is compact and every point in X is isolated, then $\mathcal{C} = \{\{x\} \mid x \in x\}$ is an open cover of X. By compactness it must have a finite subcover, but since non of the sets in this cover intersect, there are no subcovers except $\mathcal{C}$ itself. So $\mathcal{C}$ is finite and therefore X is finite.

c) Let $A \subset \mathbb{R}^n$ and let $\mathcal{B}$ be a countable basis of $\mathbb{R}^n$. If $x \in A$ is an isolated point, then $\{x\}$ is open in A. So there is an open set $U \subset \mathbb{R}^n$ such that $U \cap A = \{x\}$. There must be a basis element $B \in \mathcal{B}$ with $x \in B \subset U$ and hence $B \cap A = \{x\}$. Since there are only countably many basis elements B, only countably many points $x \in A$ can be written this way.

Exercise 6. Recall the definition of the Cantor set C from Sheets 1 and 4. Show that

a) C is compact,

b) C is metrizable,

c) C has no *isolated points*, that is no one–point subset $\{x\} \subset C$ is open,

d) Equip the set $\{0,1\}^{\mathbb{N}}$ with the *infinite product topology*, the coarsest topology which makes all projections

$$\pi_i \colon \{0,1\}^{\mathbb{N}} \to \{0,1\}, \quad (x_n)_{n\in\mathbb{N}} \mapsto x_i$$

continuous. Here $\{0,1\}$ carries the discrete topology. Show that this makes the map $\{0,1\}^{\mathbb{N}} \to C$ from Exercise Sheet 1 a homeomorphism.

Solution 6.

a) C is bounded (contained in $[0,1]$) and closed because it is the intersection of closed subsets (see the hint on Sheet 4). So C is compact by the Heine–Borel theorem.

b) C is a subset of the metric space $[0,1]$ and therefore metrizable.

c) Let $g \colon \{0,1\}^{\mathbb{N}} \to C$ be the parametrization of C from the first homework sheet. Let $x \in C$. Then $x = g((a_i)_{i\in\mathbb{N}})$ for some sequence (a_i). Let $x^n = g((b_i)) \in C$ where the sequence (b_i) differs from (a_i) only in the n–th position. Then $d(x, x^n) = 2 \cdot 3^{-n}$. So $x^n \to x$. This contradicts x being an isolated point.