Solutions 7

Exercise 1. Let X be a topological space and (x_n) a sequence in X.

a) Show that if there is an $x \in X$ such that every subsequence of (x_n) has a subsequence converging to x, then (x_n) converges to x.

b) Find a counterexample to the following statement: if every subsequence of (x_n) has a converging subsequence, then (x_n) converges.

Solution 1.

a) If (x_n) did not converge to x, there would be a neighborhood U of x such that, for every $N \in \mathbb{N}$, there is $n \geq N$ with $x_n \notin U$. We can use this to define a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ which is outside of U: first, there eixsts $n_1 \geq 1$ with $x_{n_1} \notin U$. Then, after $n_1, \ldots, n_k$ are already defined, there exists $n_{k+1} \geq n_k + 1$ with $x_{n_{k+1}} \notin U$. Now by assumption a subsequence of (x_{n_k}) converges to x. But that is impossible.

b) Consider the sequence $x_n = (-1)^n$ in $\mathbb{R}$. Every subsequence of it has a subsequence which is either constant 1 or -1.

Exercise 2. Let d and d' be two metrics on a set X, and $C \geq 1$ a constant such that

$$\frac{1}{C}\, d(x,y) \leq d'(x,y) \leq C\, d(x,y)$$

for all $x, y \in X$. Show that both metrics induce the same topology on X.

Solution 2. Let $B_\epsilon(x)$ denote the ϵ–ball in the metric d and $B'_\epsilon(x)$ the ϵ–ball in the metric d'. We want to show that the metric induced by d' is finer than the one induced by d. To this end, let $U \subset X$ be a set which is open in d, and $x \in U$. Since the open balls in d form a basis of the topology, there exists $y \in X$ and $\epsilon > 0$ with $x \in B_\epsilon(y) \subset U$. Let $\delta = \epsilon - d(x,y) > 0$. then $x \in B_\delta(x) \subset B_\epsilon(y) \subset U$. Now since $d(x,y) \leq C d'(x,y)$ for all $x, y \in X$ we have

$$B'_{\delta/C}(x) = \{z \in X \mid d'(x,z) < \delta/C\} \subset \{z \in X \mid d(x,z) < \delta\} = B_\delta(x),$$

so $x \in B'_{\delta/C}(x) \subset U$. Since this works for every $x \in U$, we can write U as a union of balls in d', so U is open in d'.

This shows that the topology induced by d' is finer than the one induced by d. The same argument with the roles of d and d' interchanged also shows that the topology from d is finer than the one from d', so the topologies are equal.

Exercise 3. Let I be some (possibly infinite) index set, and X_i a topological space for every $i \in I$. We consider the Cartesian product

$$X = \prod_{i \in I} X_i.$$

The *product topology* on $\prod_{i \in I} X_i$ is defined to be the coarsest topology which makes all the projection maps $\pi_i \colon X \to X_i$ for $i \in I$ continuous. As a convention, we write x_i for $\pi_i(x)$.

a) Show that the sets of the form $\prod_{i \in I} U_i$ with $U_i \subset X_i$ open do *not* in general form a basis of a product topology, but generate a finer topology. What would be a basis of the product topology?

b) Assume that every X_i is metrizable with a metric d_i. For every $\epsilon > 0$, $x \in X$, and every finite subset $J \subset I$ let

$$B_\varepsilon^J(x) = \{y \in X \mid d_i(x_i, y_i) < \varepsilon \; \forall i \in J\}.$$

Show that the set of all such $B_\varepsilon^J(x)$ (for all ε, J, and x) are a basis of the product topology on X.

c) Now assume in addition that $I = \mathbb{N}$. Define

$$d(x, y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{d_i(x_i, y_i)}{1 + d_i(x_i, y_i)},$$

with $x, y \in X$. Show that d is a metric.

d) Show that for every $\varepsilon > 0$ and $J \subset I$ finite there exists $\delta > 0$ with $B_\delta(x) \subset B_\varepsilon^J(x)$. (Here $B_\delta(x)$ denotes the δ–ball in the metric d.)

e) Show that for every $\varepsilon > 0$ there exists a finite $J \subset I$ and $\delta > 0$ such that $B_\delta^J(x) \subset B_\varepsilon(x)$.

f) Deduce that the metric topology on X induced by d is equal to the product topology.

g) As an example for a product of uncountably many factors consider the set

$$X = \mathbb{R}^{\mathbb{R}} = \{f \mid f \text{ is a function } f \colon \mathbb{R} \to \mathbb{R}\}.$$

We equip it with the product topology, i.e. the coarsest topology such that the projection maps

$$\pi_x \colon \mathbb{R}^{\mathbb{R}} \to \mathbb{R}, \quad f \mapsto f(x)$$

are continuous for every $x \in \mathbb{R}$. Show that $\mathbb{R}^{\mathbb{R}}$ is not first countable and hence not metrizable.

Solution 3.

a) A subbasis of the product topology is given by the sets $\pi_i(U)$, with $U \subset X_i$ open. This is because such a set has to be open in the product topology to make π_i continuous, and the topology generated by these sets is the coarsest topology to do this, and therefore equal to the product topology. We get a basis by taking all finite intersections of such sets, i.e. sets of the form $\prod_{i\in I} U_i$ with $U_i \subset X_i$ open and $U_i = X_i$ for all but finitely many i.

   Unless all but finitely many of the X_i carry the indiscrete topology, there is a set of the form $\prod_i U_i$, $U_i \subset X_i$ open, which does not contain any of these basis sets, so these generate a strictly finer topology.

b) It is easy to see that

$$B_\varepsilon^J(x) = \bigcap_{i\in J} \pi_i^{-1}(B_\varepsilon(\pi_i(x))).$$

   This is a finite intersection, so $B_\varepsilon^J(x)$ is open in the product topology. On the other hand, if $U \subset X$ is open and $x \in X$ then

$$x \in \bigcap_{i\in J} \pi_i^{-1}(U_i) \subset U$$

   for some finte set $J \subset I$ and some collection of open sets $U_i \subset X$, since the sets $\pi_i^{-1}(U_i)$ form a subbasis of the product topology. Since $\pi_i(x) \in U_i$ and J is finite, we can find $\varepsilon > 0$ such that $B_\varepsilon(\pi_i(x)) \subset U_i$ for all $i \in J$. So we get

$$x \in B_\varepsilon^J(x) \subset \bigcap_{i\in J} \pi_i^{-1}(U_i) \subset U,$$

   which shows that the sets $B_\varepsilon^J(x)$ form a basis.

c) First of all, the series defining d always converges, since

$$0 \leq \frac{d_i(x_i, y_i)}{1 + d_i(x_i, y_i)} < 1$$

   so the sequence of partial sums is non–decreasing and bounded above since

$$\sum_{i=1}^{\infty} \frac{1}{2^i} \frac{d_i(x_i, y_i)}{1 + d_i(x_i, y_i)} \leq \sum_{i=1}^{\infty} \frac{1}{2^i} = 1.$$

   Furthermore, since all summands are non–negative, $d(x, y) = 0$ implies that $d_i(x_i, y_i) = 0$ for all $i \in I$. Symmetry is clear and transitivity follows from the monotonicity of positive linear combinations and the fact that

$$a \leq b + c \quad \Rightarrow \quad \frac{a}{1+a} \leq \frac{b}{1+b} + \frac{c}{1+c} \quad \forall a, b, c \geq 0$$

   which is elementary to prove by rearranging the latter inequality.

d) Let $\varepsilon > 0$ and $J \subset \mathbb{N}$ finite. Define

$$\delta = \min_{i \in J} \frac{\varepsilon}{2^i(1+\varepsilon)}.$$

Then if $y \in B_\delta(x)$, then $d(x,y) < \delta$, so for every $j \in J$

$$\frac{1}{2^j}\frac{d_j(x_j,y_j)}{1+d_j(x_j,y_j)} \leq \sum_{i=1}^{\infty} \frac{1}{2^i}\frac{d_i(x_i,y_i)}{1+d_i(x_i,y_i)} = d(x,y) < \delta \leq \frac{1}{2^j}\frac{\varepsilon}{1+\varepsilon}.$$

Using the strict monotonicity of the function $x \mapsto \frac{x}{1+x}$ we get $d_j(x_j,y_j) < \varepsilon$ for every $j \in J$, i.e. $y \in B^J_\varepsilon(x)$.

e) Let $\varepsilon > 0$. We can assume wlog that $\varepsilon < 1$. Find $k \in \mathbb{N}$ such that $2^{-k} \leq \varepsilon/2$. Set $J = \{1, \dots, k\}$ and

$$\delta = \frac{\varepsilon/2}{1 - 2^{-k} - \varepsilon/2} > 0,$$

so that $\frac{\delta}{1+\delta} = \frac{\varepsilon/2}{1-2^{-k}}$. Now if $y \in B^J_\delta(x)$, then $d_i(x_i,y_i) < \delta$ for all $i \in J$. Therefore

$$\begin{aligned} d(x,y) &= \sum_{i=1}^{k} \frac{1}{2^i}\frac{d_i(x_i,y_i)}{1+d_i(x_i,y_i)} + \sum_{i=k+1}^{\infty} \frac{1}{2^i}\frac{d_i(x_i,y_i)}{1+d_i(x_i,y_i)} \\ &< \sum_{i=1}^{k} \frac{1}{2^i} \cdot \frac{\delta}{1+\delta} + \sum_{i=k+1}^{\infty} \frac{1}{2^i} \cdot 1 = (1-2^{-k})\frac{\delta}{1+\delta} + 2^{-k} \leq \varepsilon, \end{aligned}$$

so $y \in B_\varepsilon(x)$. Note that in the second step we used that the function $x \mapsto \frac{x}{1+x}$ is strictly increasing.

f) This follows from d) and e).

To be precise, if $y \in B^J_\varepsilon(x)$, then $B^J_{\varepsilon'}(y) \subset B^J_\varepsilon(x)$ for some ε', and by d) there exists $\delta > 0$ such that $B_\delta(y) \subset B^K_{\varepsilon'}(y)$. Doing this for all $y \in B^J_\varepsilon(x)$ and taking the union expresses $B^J_\varepsilon(x)$ as a union of open balls in the metric d, so it is open in the metric topology.

By essentially the same argument, every metric ball $B_\varepsilon(x)$ is also open in the product topology. Together, this shows that the toplogies are equal.

g) For every $x \in \mathbb{R}$ define the set

$$U_x = \pi_x^{-1}((0,\infty)) = \{f \in \mathbb{R}^\mathbb{R} \mid f(x) > 0\}.$$

All of them contain the constant function $f = 1$. Now assume that $\mathbb{R}^\mathbb{R}$ was second countable. Then this function f has a countable neighborhood basis $(B_i)_{i \in \mathbb{N}}$. This means that for every $x \in \mathbb{R}$ we can choose an $i_x \in \mathbb{N}$ such that $f \in B_{i_x} \subset U_x$. For every $k \in \mathbb{N}$ define $R_k = \{x \in \mathbb{R} \mid i_x = i\}$. Then

$$B_k \subset \bigcap_{x \in R_k} U_x.$$

But by the definition of the product topology, the open set $B_k \subset \mathbb{R}^{\mathbb{R}}$ satisfies $\pi_x(B_k) = \mathbb{R}$ for all but finitely many $x \in \mathbb{R}$. This means that R_k must be finite for every $k \in \mathbb{N}$. But the union of all R_k is $\mathbb{R}$, which is uncountable. This is a contradiction.