Solutions 8

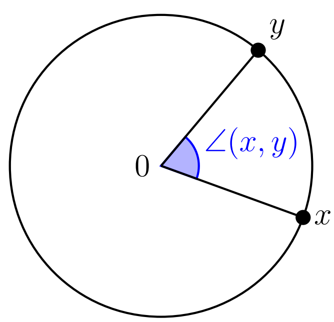
Exercise 1. Consider the circle

$$S^1 = \{x \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = 1\}.$$

For two points $x, y \in S^1$, let d be their Euclidean distance

$$d(x,y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$$

and $\angle(x,y)$ the angle between the two points as seen from the origin (as a number in $[0,\pi]$).



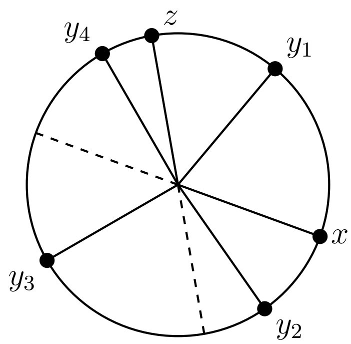
Show that d and $\angle$ both define metrics on S^1 and both these metrics induce the subspace topology on S^1 (as a subset of $\mathbb{R}^2$).

Solution 1. More generally, if (X,d) is a metric space and $A \subset X$, then we can define a metric d_A on A by restricting d, i.e.

$$d_A(x,y) = d(x,y) \quad \forall x, y \in A.$$

All the metric axioms clearly carry over. Furthermore, if $x \in A$, then $B_\epsilon^{d_A}(x) = B_\epsilon^d(x) \cap A$ and therefore the topology generated by the d_A–balls is the subspace topology on A.

To show that $\angle$ is a metric, we just need to check the axioms. It is clear that $\angle(x,y) = \angle(y,x)$ and that $\angle(x,y) = 0$ if and only if $x = y$. To check the triangle inequality, just look at the different cases how angles can add up, as seen in this picture:



You see that

$$\begin{aligned}
&\angle(x,z) = \angle(x,y_1) + \angle(y_1,z),\\
&\angle(x,z) \le \angle(y_2,z),\\
&\angle(x,z) \le \pi \le \angle(x,y_3) + \angle(y_3,z),\\
&\angle(x,z) \le \angle(x,y_4).
\end{aligned}$$

The triangle inequality is satisfied in any case.

Finally, this metric induces the same topology as d. In fact, using some trigonometry we even get that

$$d(x,y) = 2\sin\frac{\angle(x,y)}{2},$$

so every open ball in d is an open ball in $\angle$ and vice–versa, just with a different radius.

Exercise 2. Let X be a metric space with metric d and $K \subset U \subset X$ with K compact and U open. Show that there exists $\varepsilon > 0$ with $N_\varepsilon(K) \subset U$, where

$$N_\varepsilon(K) = \{x \in X \mid \operatorname{dist}(x,K) < \varepsilon\}.$$

Solution 2. Let $x \in K$. Since U is open, there is an $\varepsilon_x > 0$ such that $B_{\varepsilon_x}(x) \subset U$. The collection of balls half that radius, i.e. $\{B_{\varepsilon_x/2}(x) \mid x \in K\}$ is an open cover of K, so it has a finite subcover. This means there are $x_1,\dots,x_n$ and $\varepsilon_1,\dots,\varepsilon_n$ such that

$$K \subset B_{\varepsilon_1/2}(x_1) \cup \cdots \cup B_{\varepsilon_n/2}(x_n) \subset U.$$

Let $\varepsilon = \min\{\varepsilon_1/2,\dots,\varepsilon_n/2\}$. Then for every $y \in N_\varepsilon(K)$ we find $z \in K$ with $d(y,z) = \operatorname{dist}(y,K) < \varepsilon$ (1c from homework 6, though something weaker than that would suffice). Since $z \in B_{\varepsilon_i/2}(x_i)$ for some i, $d(y,x_i) \le d(y,z) + d(z,x_i) < \varepsilon + \varepsilon_i/2 \le \varepsilon_i$, hence $y \in B_{\varepsilon_i}(x_i) \subset U$. So we get $N_\varepsilon(K) \subset U$.

Exercise 3. Let X be a compact space and Y a metric space with metric d. Let $C(X,Y)$ be the set of all continuous functions from X to Y.

a) Show that
$$d_C(f,g) = \sup_{x\in X} d(f(x),g(x))$$
defines a metric d_C on $C(X,Y)$.

b) For a compact subset $K \subset X$ and an open subset $U \subset Y$ define
$$V_{K,U} = \{f \in C(X,Y) \mid f(K) \subset U\}.$$
Show that the sets $V_{K,U}$ are open in the metric d_C.

c) Show that
$$\mathcal{S} = \{V_{K,U} \mid K \subset X \text{ compact and } U \subset Y \text{ open}\}$$
is a subbasis of the metric topology on $C(X,Y)$ with the metric d_C.

Hint: You will need to show that $f \in V_{K_1,U_1} \cap \cdots \cap V_{K_n,U_n} \subset B_\varepsilon(f)$ for some suitable choices of $K_1,\ldots,K_n$ and $U_1,\ldots,U_n$. Choose the K_i so that they cover X but every one of them is small. Then choose the U_i a little bit bigger than $f(K_i)$.

d) Conclude that the topology on $C(X,Y)$ does not depend on the metric on Y, but only the topology on Y. It is called the *compact open topology*, and can even be defined using $\mathcal{S}$ without assuming that Y is metrizable.

Solution 3.

a) Symmetry and the triangle inequality directly carry over. If $d_C(f,g) = 0$, then $d(f(x),g(x)) = 0$ for all $x \in X$ and hence $f(x) = g(x)$.

b) Let $f \in V_{K,U}$. We want to show that $B_\varepsilon(f) \subset V_{K,U}$ for some $\varepsilon > 0$. Here $B_\varepsilon(f)$ is a ball in the metric d_C. By Exercise 2, there is an $\varepsilon > 0$ with $N_\varepsilon(f(K)) \subset U$. We claim that for all $g \in B_\varepsilon(f)$ we have $g(K) \subset N_\varepsilon(f(K))$, so $g(K) \subset U$ and therefore $g \in V_{K,U}$. Indeed, if $x \in K$, then $d(g(x),f(x)) \le d_C(g,f) < \varepsilon$, so $g(x) \in B_\varepsilon(f(x)) \subset N_\varepsilon(f(K))$.

c) Denote by $\mathcal{T}$ the topology gerated by $\mathcal{S}$ and by $\mathcal{T}'$ the topology induced from d_C. By b) we already know that $\mathcal{T} \subset \mathcal{T}'$, so we only need to show $\mathcal{T}' \subset \mathcal{T}$. It suffices to show that for every $f \in C(X,Y)$ and $\varepsilon > 0$ that there is an open set $\mathcal{U} \in \mathcal{T}$ with $f \in \mathcal{U} \subset B_\varepsilon(f)$.

We will produce such a $\mathcal{U}$ as an intersection of finitely many sets $V_{K_1,U_1},\ldots,V_{K_n,U_n}$. We want to choose some small compact sets $K_1,\ldots,K_n \subset X$ which together cover

X. We will determine later how to choose them. Then we also set $U_i = N_{\varepsilon/2}(f(K_i))$. By definition $f \in V_{K_i,U_i}$ for all i.

To show that their intersection is contained in $B_\varepsilon(f)$, let $g \in \bigcap_i V_{K_i,U_i}$, and $x \in X$ be arbitrary. Then $x \in K_i$ for some i (because the K_i cover X), and $g(x) \in g(K_i) \subset U_i = N_\varepsilon(f(K_i))$. So there is some $y \in K_i$ with $d(f(y), g(x)) < \varepsilon/2$. We would like to also have $d(f(x), f(y)) < \varepsilon/2$, so that we can conclude $d(f(x), g(x)) < \varepsilon$ and hence $d_C(f, g) < \varepsilon$ as x was arbitrary.

So how should we choose the K_i to get the inequality $d(f(x), f(y)) < \varepsilon/2$ for all $x, y \in K_i$? For every $x \in X$ the set $f^{-1}(B_{\varepsilon/4}(f(x)))$ is an open neighborhood of x, so these sets cover X. Since X is compact, finitely many of these sets already cover X. We choose these as our K_i, i.e.

$$K_i = f^{-1}(B_{\varepsilon/4}(f(x_i)))$$

for some points $x_1, \ldots, x_n$. If $x, y \in K_i$, then $d(f(x), f(y)) \leq d(f(x), f(x_i)) + d(f(y), f(x_i)) < \varepsilon/4 + \varepsilon/4 = \varepsilon/2$.

d) We have shown that the topology induced by d_C can be described by $\mathcal{S}$, which does not depend on the metric on Y, but only on the notions of compact set in X and open set in Y, i.e. the topologies on X and Y.

Exercise 4. Let (X, d_X) and (Y, d_Y) be metric spaces. A map $f\colon X \to Y$ is called *uniformly continuous* if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $d_X(x_1, x_2) < \delta$ implies $d_Y(f(x_1), f(x_2)) < \varepsilon$ for all $x_1, x_2 \in X$.

a) Show that every uniformly continuous map is continuous.

b) Show that the converse holds if X is compact, i.e. every continuous map is uniformly continuous.

Hint: Given ε the continuity gives you a δ for every $x \in X$. We just have to show that a single δ works for every x. It might be helpful to use the following fact which we proved as part of Theorem 3.6: if $\mathcal{C}$ is an open cover of a compact space X, then there exists an $r > 0$ such that every ball of radius r is contained in a single set $U \in \mathcal{C}$ of the cover.

c) Give an example of a map which is continuous but not uniformly continuous.

Solution 4.

a) Let $x \in X$ and $\varepsilon > 0$. Uniform continuity tells us that there is a $\delta > 0$ such that $d_X(x_1, x_2) < \delta$ implies $d_Y(f(x_1), f(x_2)) < \varepsilon$ for all $x_1, x_2 \in X$. In particular $d_X(x, x') < \delta$ implies $d_Y(f(x), f(x')) < \varepsilon$ for all $x' \in X$, so f is continuous.

b) Let $\varepsilon > 0$. Since f is continuous, for every $x \in X$ there is a $\delta_x > 0$ such that $d_X(x, x') < \delta$ implies $d_Y(f(x), f(x')) < \varepsilon/2$. Now if $x_1, x_2 \in B_{\delta_x}(x)$, then

$$d(f(x_1), f(x_2)) \leq d(f(x_1), f(x)) + d(f(x_2), f(x)) < \varepsilon.$$

The sets $B_{\delta_x}(x)$ for all $x \in X$ form an open cover of X. Since X is compact, there is $r > 0$ such that every r–ball is contained in one of these sets. Whenever we have $x_1, x_2 \in X$ with $d(x_1, x_2) < r$, then they are contained in a common r–ball, so they are both in $B_{\delta_x}(x)$ for some $x \in X$, which implies that $d(f(x_1), f(x_2)) < \varepsilon$. So f is uniformly continuous.

c) An example is the function $x \mapsto 1/x$ on the positive reals. If it was uniformly continuous there would be a $\delta > 0$ such that $|1/x - 1/y| < 1$ whenever $|x - y| < \delta$. But if e.g. we choose $x, y \in (0, \infty)$ so that $x - y = xy = \delta/2$ (that is, x and y are $\sqrt{\delta^2/16 + \delta/2} \pm \delta/4$), then $|x - y| = \delta/2 < \delta$ and $|1/x - 1/y| = |\frac{x-y}{xy}| = 1$, a contradiction.