Solutions 9

Definition 1. A map is called *open* if the image of every open set is open, and *closed* if the image of every closed set is closed.

Definition 2. A group action $G \times X \to X$ on a topological space X is *continuous* if for every $g \in G$ the map $X \to X, x \mapsto gx$ is continuous.

Exercise 1. Let X, Y be topological spaces and $f\colon X \to Y$ a continuous surjective map. Show that

a) If f is a quotient map, then $Y \cong X/\sim$ where $\sim$ is the equivalence relation defined by $x \sim x' \Leftrightarrow f(x) = f(y)$.

b) If f is open, then f is a quotient map.

c) If f is closed, then f is a quotient map.

d) Find an example of a quotient map which is neither open nor closed.

Solution 1.

a) As we have proved as Lemma 4.2, there exists a continuous function $g\colon X/\sim \to Y$ such that $g \circ p = f$, where $p\colon X \to X/\sim$ is the natural projection. We want to showt that this map is in fact a homeomporphism. To see that it is injective, let $x, y \in X$ and assume that $g([x]) = g([y])$. Then $f(x) = f(y)$, so $x \sim y$ and hence $[x] = [y]$. To see that g is surjective, let $y \in Y$. Since f is surjective there is $x \in X$ with $f(x) = y$ and therefore $g([x]) = f(x) = y$.

   It remains to show that g is an open map. Let $U \subset X/\sim$ be open. Then $g(U) = f(p^{-1}(U))$ and since f is a quotient map this is open if and only if $f^{-1}(f(p^{-1}(U)))$ is open. We claim that this set is equal to $p^{-1}(U)$ and hence open. Indeed, if $x \in p^{-1}(U)$, then $f(x) \in f(p^{-1}(U))$, so $x \in f^{-1}(f(p^{-1}(U)))$. Conversely, if $x \in f^{-1}(f(p^{-1}(U)))$, then $f(x) \in f(p^{-1}(U))$ which means that $f(x) = f(x')$ for some x' with $p(x') \in U$. But by definition of the equivalence relation, $x \sim x'$ and therefore $p(x) = p(x') \in U$, so $x \in p^{-1}(U)$ as claimed.

b) If f is open and $U \subset Y$ such that $f^{-1}(U)$ is open, then $f(f^{-1}(U))$ is open. But this is U since f is surjective, so f is a quotient map.

c) If f is closed and $U \subset Y$ such that $f^{-1}(U)$ is open, then $f(X \setminus f^{-1}(U))$ is closed. Again since f is surjective, $f(X \setminus f^{-1}(U)) = Y \setminus U$, so U is open and f is a quotient map.

d) One of many possible examples would be this variation of the line with two origins: take $X = \mathbb{R} \times \{0, 1\}$ and the equivalence relation $(x, 0) \sim (x, 1)$ if and only if $x \in [0, 1)$.

   If $p\colon X \to X/{\sim}$ is the natural projection, then $\mathbb{R} \times \{0\}$ is open and closed in X, but

$$p^{-1}(p(\mathbb{R} \times \{0\})) = (\mathbb{R} \times \{0\}) \cup ([0, 1) \times \{1\}),$$

   which is neither open nor closed. So $p(\mathbb{R} \times \{0\})$ is also neither open nor closed.

Exercise 2. Let $G \times X \to X$ be a continuous group action. Show that the projection $p\colon X \to X/G$ is open.

Solution 2. Let $U \subset X$ be open. We want to show that $p(U)$ is open, which by definition means showing that $p^{-1}(p(U))$ is open. But

$$p^{-1}(p(U)) = \{x \in X \mid \exists g \in G\colon gx \in U\} = \bigcup_{g \in G} g^{-1}(U),$$

where we view g as a function $g\colon X \to X, x \mapsto gx$. This function is continuous by assumption, so $g^{-1}(U)$ is open for every $g \in G$. Since $p^{-1}(p(U))$ is a union of open sets, it is open.

Exercise 3. Let $G \times X \to X$ be a continuous group action on a topological space, $\sim$ the associated equivalence relation, and $p\colon X \to X/G$ the natural projection. Further let $A \subset X$ be a subset with the property that $p(A) = p(X)$.

a) Show that $A/{\sim} \cong X/G$ (the symbol $\cong$ means *is homeomorphic to*).

b) Show that

$$I/{\approx} \cong \mathbb{R}/\mathbb{Z},$$

   where $\approx$ is the equivalence relation defined by $0 \approx 1$ and $\mathbb{Z}$ acts on $\mathbb{R}$ by addition.

c) Show that

$$I^2/{\simeq} \cong \mathbb{R}^2/\mathbb{Z}^2,$$

   where $\simeq$ is defined by $(0, x) \simeq (1, x)$ and $(x, 0) \simeq (x, 1)$ for all $x \in I$, and the group $(\mathbb{Z}^2, +)$ acts on $\mathbb{R}^2$ by (vector) addition.

Solution 3.

a) Restricting the map p to A gives a map $p|_A\colon A \to X/G$ which is still continuous and surjective. It has the property that $p(x) = p(y)$ whenever $x \sim y$, so by Lemma 4.2 there is a map $f\colon A/{\sim} \to X/G$ which is continuous and satisfies $f([x]) = [x]$ for all $x \in A$. By the assumption $p(A) = p(X)$ every equivalence class has a representative in A, so f is surjective. Also f is clearly injective. To see that f is open, let $U \subset A/{\sim}$ be open. The preimage $\pi^{-1}(U)$ under the natural projection $\pi\colon A \to A/{\sim}$ is open, and $p|_A\colon A \to X/G$ is an open map by Exercise 2, so $f(U) = p|_A(\pi^{-1}(U))$ is open.

b) We just need to check all the assumptions to apply part a). The equivalence relation coming from the group action of $\mathbb{Z}$ on $\mathbb{R}$, when restricted to I, indeed only identifies 0 with 1, and every equivalence class has a representative in the unit interval.

c) As in b) we have to check that $\simeq$ is the equivalence relation coming from the group action of $\mathbb{Z}^2$ on $\mathbb{R}^2$ restricted to I^2, and that every equivalence class has a representative in I^2. Then we can apply part a).

Exercise 4. Let $R > r > 0$. Show that the map

$$f\colon \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{R}^3, \quad [(s,t)] \mapsto \begin{pmatrix} R\cos(2\pi s) + r\sin(2\pi t)\cos(2\pi s) \\ R\sin(2\pi s) + r\sin(2\pi t)\sin(2\pi s) \\ r\cos(2\pi t) \end{pmatrix}$$

is well–defined and an imbedding and that its image is the set $(x,y,z) \in \mathbb{R}^3$ solving the equation

$$\left(\sqrt{x^2+y^2} - R\right)^2 + z^2 = r^2.$$

Convince yourself that this is indeed the torus.

Solution 4. Consider the map

$$\widetilde{f}\colon \mathbb{R}^2 \to \mathbb{R}^3, \quad (s,t) \mapsto \begin{pmatrix} R\cos(2\pi s) + r\sin(2\pi t)\cos(2\pi s) \\ R\sin(2\pi s) + r\sin(2\pi t)\sin(2\pi s) \\ r\cos(2\pi t) \end{pmatrix}.$$

By Lemma 4.2 there is a continuous map $f\colon \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{R}^3$ with $f \circ p = \widetilde{f}$, where $p\colon \mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2$ is the natural projection.

To see that f is an imbedding, we need to show that it is injective and a homeomorphism onto its image. We can show the latter by hand, but a handy trick is the following: We know that $\mathbb{R}^2/\mathbb{Z}^2$ is compact, since by Exercise 3 it is homeomorphic to the image of the compact space I^2 under a continuous projection. Futher the image $f(\mathbb{R}^2/\mathbb{Z}^2)$

as a subspace of $\mathbb{R}^3$ is Hausdorff, so Exercise 4b of Homework 5 proves that f is a homeomorphism onto its image.

To see that f is injective, assume we have $s, t, s', t' \in \mathbb{R}$ with $\widetilde{f}(s,t) = \widetilde{f}(s',t')$. Then, by adding the squares of the first two coordinates,

$$\begin{aligned}(R + r\sin(2\pi t))^2 &= (R\cos(2\pi s) + r\sin(2\pi t)\cos(2\pi s))^2 + (R\sin(2\pi s) + r\sin(2\pi t)\sin(2\pi s))^2 \\ &= (R\cos(2\pi s') + r\sin(2\pi t')\cos(2\pi s'))^2 + (R\sin(2\pi s') + r\sin(2\pi t')\sin(2\pi s'))^2 \\ &= (R + r\sin(2\pi t'))^2\end{aligned}$$

By the assumption $R > r > 0$ we know that $R + r\sin(2\pi t), R + r\sin(2\pi t') > 0$, so $\sin(2\pi t) = \sin(2\pi t')$. We also have $\cos(2\pi t) = \cos(2\pi t')$, so $t' = t + k$ for some $k \in \mathbb{Z}$. Furthermore, dividing by $R + r\sin(2\pi t) = R + r\sin(2\pi t')$ in the first two coordinates gives us $s' = s + l$ for some $l \in \mathbb{Z}$. So $[(s,t)] = [(s',t')]$, i.e. f is injective.

We have shown that f is an imbedding. Now we still need to see that its image is indeed the torus. For one inclusion, just plug

$$\begin{aligned}x &= R\cos(2\pi s) + r\sin(2\pi t)\cos(2\pi s) \\ y &= R\sin(2\pi s) + r\sin(2\pi t)\sin(2\pi s) \\ z &= r\cos(2\pi t)\end{aligned}$$

into the expression $(\sqrt{x^2+y^2} - R)^2 + z^2$ and verify that it always gives r^2. For the other inclusion, let (x,y,z) be any solution of that equation, and define

$$\begin{aligned}u &= \frac{z}{r} \\ v &= \frac{\sqrt{x^2+y^2} - R}{r} \\ p &= \frac{x}{R + rv} \\ q &= \frac{y}{R + rv}\end{aligned}$$

Then $u^2 + v^2 = 1$ and $p^2 + q^2 = \frac{x^2+y^2}{(R+rv)^2} = \frac{x^2+y^2}{x^2+y^2} = 1$. Note that in particular $v \geq -1$, and therefore $R + rv \neq 0$, so p and q are always defined. So we can write $u = \cos(2\pi t)$, $v = \sin(2\pi t)$, $p = \cos(2\pi s)$ and $q = \sin(2\pi s)$ for some $s, t \in \mathbb{R}$. Now

$$f([s,t]) = \begin{pmatrix} R\cos(2\pi s) + r\sin(2\pi t)\cos(2\pi s) \\ R\sin(2\pi s) + r\sin(2\pi t)\sin(2\pi s) \\ r\cos(2\pi t)\end{pmatrix} = \begin{pmatrix} Rp + rvp \\ Rq + rvq \\ ru \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

so the image of f hits every point (x,y,z) on the torus.