Solutions 10

Exercise 1 (Proof writing exercise, due April 28)**.** To practice your proof writing skills, pick one of the longer exercises from the past homework and write up your solution again, as clearly and detailed as possible. Even if your solution got all points, that only means it's correct. But it is always possible to improve the exposition!

- Pick your favorite one of the following exercises (sheet.exercise): 7.3, 8.3 or 9.4.
- Fix any mathematical mistakes, if they existed in your original solution.
- Critically review your write–up, add missing details and improve sections which are not completely clear.
- Actually, it can be better to start over and write everything another time. This makes sure that you have really revisited every sentence. Maybe you find a simpler argument for some part, then use that instead.
- The goal is to end up with a proof which would be understandable for *everyone* else in the class, even if they didn't try to solve the problem themselves.
- If you think your solution meets this requirement, try to improve it even further. If a section is very technical and hard to read, maybe explain the rough idea first. Maybe add pictures. Do whatever you think makes it easier to understand.
- When you're done, find a partner and let them read your write–up. Incorporate their suggestions (if you agree with them). Indicate who read your solution on the submission.
- Finally, submit your proof on Canvas until April 28.
- This counts outside of the usual homework, and will make up a higher percentage of your final grade. Other than the usual homeworks, this is only about form, and while your argument should still be correct, that's not the main focus.

We also have some "normal" exercises this week, but it's slightly shorter than usual.

Definition 1. Let $f, g\colon X \to Y$ be homeomorphisms of topological spaces X and Y. They are said to be *isotopic* if there is a continuous family $f_t\colon X \to Y$ of maps for $t \in I$ such that $f = f_0$ and $g = f_1$, and f_t is a homeomorphism for every $t \in I$. Here "continuous family" means that the map

$$I \times X \to Y, \quad (t, x) \mapsto f_t(x)$$

is continuous.

Exercise 2. Show that every homeomorphism $f\colon [0,1] \to [0,1]$ is isotopic to either the identity or the map $t \mapsto 1 - t$.

Solution 2. 0 and 1 are the only points in $[0,1]$ whose complement is connected. Since this property is preserved by the homeomorphism f, we have $\{f(0), f(1)\} = \{0,1\}$. We can assume that $f(0) = 0$ and $f(1) = 1$, otherwise consider the function $1 - f$.

Define $f_t\colon I \to I$ by $f_t(x) = (1-t)x + tf(x)$. This is a continuous family of maps with $f_0 = \mathrm{id}$ and $f_t = f$. Furthermore, $f_t(0) = 0$ and $f_t(1) = 1$ for all $t \in I$. So by the intermediate value theorem f_t is surjective for every t. If f_t was not injective, there would be $x < y$ with

$$(1-t)x + tf(x) = (1-t)y + tf(y).$$

This implies $f(x) = f(y) + t^{-1}(1-t)(y-x) > f(y)$. By the intermediate value theorem, there exists $z \in (0, x)$ with $f(z) = f(y)$, so $y = z$ since f is injective. However, this contradicts $x < y$.

Now every f_t is a bijective continuous map from I to itself. As I is compact and Hausdorff, Exercise 4b from Sheet 5 implies that f_t is a homeomorphism. So f is isotopic to the identity.

Exercise 3. For a homeomorphism $f\colon I \to I$ we consider the space $I^2/{\sim_f}$ where the equivalence relation $\sim_f$ is given by $(0, x) \sim_f (1, f(x))$ for all $x \in I$. In other words, the square with the two vertical edges glued together via f.

Show that if f and g are isotopic, then $I^2/{\sim_f}$ is homeomorphic to $I^2/{\sim_g}$.

Solution 3. By the assumption that f and g are isotopic, there exists a continuous family of functions $h_s\colon I \to I$ such that $h_0 = f$, $h_1 = g$, and h_s is a diffeomorphism for every $s \in I$. Now consider the map

$$\psi\colon I \times I \to I \times I, \quad (s, t) \mapsto (s, h_s(f^{-1}(t))).$$

This is continuous and bijective with inverse $(s,t) \mapsto (s, f(h_s^{-1}(t)))$. Furthermore

$$\psi(0,x) = (0, h_0(f^{-1}(x))) = (0,x), \quad \psi(1, f(x)) = (1, h_1(f^{-1}(f(x)))) = (1, g(x)),$$

so $z \sim_f z'$ if and only if $\psi(z) \sim_g \psi(z')$, for all $z, z' \in I^2$. This means that ψ defines a continuous map $\varphi\colon I^2/{\sim_f} \to I^2/{\sim_g}$ by Lemma 4.2, and that this map is still bijective. Since $I^2/{\sim_f}$ is compact, and $I^2/{\sim_g}$ is Hausdorff (because the natural projection of this quotient is a closed map, Lemma 4.5), φ is a homeomorphism (Sheet 5, Exercise 4b).

Exercise 4. Recall that the *real projective plane* $\mathbb{R}\mathrm{P}^2$ is the quotient of the action of the group $(\mathbb{R}\setminus\{0\}, +)$ on $\mathbb{R}^3\setminus\{0\}$ via scalar multiplication. In other words, if $x, y \in \mathbb{R}^3\setminus\{0\}$, then $[x] = [y] \in \mathbb{R}\mathrm{P}^2$ if and only if x and y lie on the same line through the origin in $\mathbb{R}^3$. We write

$$p\colon \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}\mathrm{P}^2, \quad x \mapsto [x]$$

for the natural projection.

In this exercise we want to show that the space obtained by polygon gluing with the labelling scheme *abab* is the real projective plane $\mathbb{R}\mathrm{P}^2$.

a) Show that the map

$$f\colon [-1,1]^2 \to D, \quad x \mapsto \frac{\max\{|x_1|, |x_2|\}}{\sqrt{x_1^2 + x_2^2}}\, x$$

is a homeomorphism, where $D = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \le 1\}$ is the unit disk.

b) Let

$$H = \{(x,y,z) \in S^2, z \ge 0\}$$

be the upper half–sphere. Show that there is a homeomorphism $g\colon D \to H$ with

$$g(x,y) = (x,y,0) \qquad \forall (x,y) \in S^1 \subset D.$$

c) Consider the composition

$$F\colon [-1,1]^2 \xrightarrow{f} D \xrightarrow{g} H \xrightarrow{\iota} \mathbb{R}^3 \setminus \{0\} \xrightarrow{p} \mathbb{R}\mathrm{P}^2.$$

where $\iota\colon H \to \mathbb{R}^3 \setminus \{0\}$ is just the inclusion map. Show that F a quotient map, and therefore $\mathbb{R}\mathrm{P}^2 \cong [-1,1]^2/{\sim}$ for the equivalence relation defined by $x \sim y \Leftrightarrow F(x) = F(y)$. Show that this equivalence relation can be described as gluing along the labelling scheme *abab*.

Hint for all parts: you can save some work by looking at Exercise 4 on Sheet 5. Also, these pictures might be helpful:

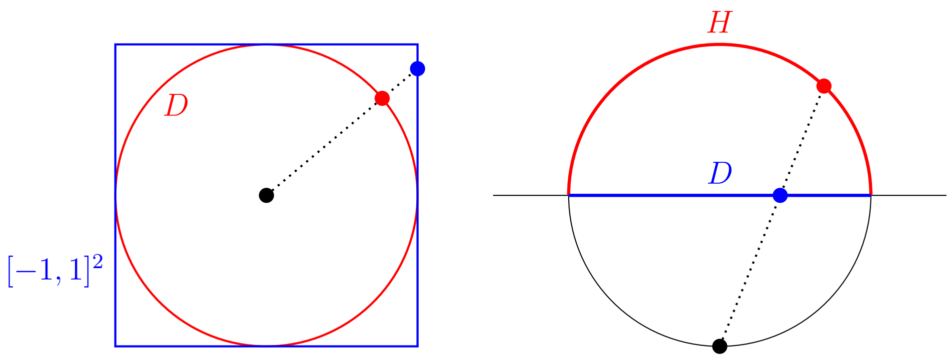

Figure 1: Pictures which might help you understand the homeomorphisms in a and b.

Solution 4.

a) By Exercise 4b) from Sheet 5, we only need to show that f is well–defined, continuous and bijective. Checking this is straightforward.

b) A possible homeomorphism (the stereographic projection, as indiciated in the figure) is given by

$$g(x,y) = \frac{1}{x^2+y^2+1}\left(2x, 2y, 1-x^2-y^2\right)$$

with the inverse

$$g^{-1}(x,y,z) = \left(\frac{x}{z+1}, \frac{y}{z+1}\right).$$

Again, we only need to show that this defines a continuous, bijective map from D to H.

c) The map F is a composition of continuous maps, so it is continuous. We know that f and g are homeomorphisms, and $p \circ \iota$ surjective since every $x \in \mathbb{R}^3 \setminus \{0\}$ is equivalent by scalar multiplication to an element of H. So the composition F is surjective.

Because f and g are homeomorphisms, to show that F is a quotient map we actually just need to show that $p \circ \iota = p|_H \colon H \to \mathbb{R}\mathrm{P}^2$ is a quotient map. It is clearly surjective and continuous, so by Sheet 9, Exercise 1c, it is enough to show that $p|_H$ is a closed map. Let $C \subset H$ be closed. Then

$$p^{-1}(p(C)) = \{\lambda x \mid x \in C, \lambda \in \mathbb{R} \setminus \{0\}\} = \{\lambda x \mid x \in C \cup -C, \lambda > 0\}.$$

Let $(y_i)_{i\in\mathbb{N}}$ be a sequence in this set converging to $y \in \mathbb{R}^3 \setminus \{0\}$. Then $\|y_i\|^{-1} y_i \in C \cup -C$ and $\|y_i\|^{-1} y_i \to \|y\|^{-1} y$. As the union of two closed sets, $C \cup -C$ is closed, so $\|y\|^{-1} y \in C \cup -C$, hence $y \in p^{-1}(p(C))$. So this set and therefore $p(C)$ is closed.

With the equivalence relation $\sim$ on $[-1,1]^2$ defined by $x \sim y \Leftrightarrow F(x) = F(y)$, the map $F' \colon [-1,1]^2/{\sim} \to \mathbb{RP}^2$ with $F'([x]) = F(x)$ is well–defined and injective, and also surjective since F is surjective. So by Sheet 5, exercise 4b) F' is a homeomorphism.

Let $x, y \in [-1,1]^2$ such that $F(x) = F(y)$. This means $g(f(x)) = \lambda g(f(y))$ for $\lambda \in \mathbb{R} \setminus \{0\}$. Clearly, this is equivalent to either $g(f(x)) = g(f(y))$ and therefore $x = y$, or $g(f(x)) = -g(f(y))$, which can only happen if the third coordinate of both is 0. The latter case is equivalent to $f(x) = -f(y) \in S^1$, and hence also to

$$x = -y \in \partial[-1,1]^2 = (\{-1,1\} \times [-1,1]) \cup ([-1,1] \times \{-1,1\}).$$

This is precisely the equivalence given by gluing along the labeling scheme *abab*.