Spring 2020 – Math 367K – 53115

Solutions 11

Exercise 1. Show that every compact locally Euclidean space is second countable.

In particular, this applies to the spaces obtained by gluing polygon sides in pairs, and completes the proof that they are manifolds.

Solution 1. Let X be compact and locally Euclidean. Then every $x \in X$ has a neighborhood U which is homeomorphic to an open subset of $\mathbb{R}^n$. These neighborhoods cover X, and since X is compact, there is a finite subcover $U_1, \ldots, U_k$. As each U_i is homeomorphic to an open subset of $\mathbb{R}^n$, it has a countable basis $\mathcal{B}_i$. Their union $\bigcup_i \mathcal{B}_i$ is still countable and is a basis of X: if $x \in V \subset X$, V open, and $x \in U_i$, then there is a basis element $B \in \mathcal{B}_i$ with $x \in B \subset V \cap U_i \subset V$.

Exercise 2. Show that the fundamental group $\pi_1(X, x_0)$ is a group. More precisely, show that, for all loops α, β, γ at x_0,

a) $(\alpha * \beta) * \gamma \simeq \alpha * (\beta * \gamma)$,

b) $c * \alpha \simeq \alpha \simeq \alpha * c$,

c) $\alpha * \overline{\alpha} \simeq c \simeq \overline{\alpha} * \alpha$.

Here c is the constant loop $c(t) = x_0$ and $\overline{\alpha}(t) = \alpha(1 - t)$.

We already proved a) in class. Make sure you understand the proof and check the details.

Solution 2. We can for example define the homotopies $h_1, h_2, h_3, h_4, h_5 \colon I \times I \to X$ by

$$h_1(s,t) = \begin{cases} \alpha\big(\frac{4t}{s+1}\big) & \text{if } 0 \le t \le \frac{s+1}{4} \\ \beta(4t-s-1) & \text{if } \frac{s+1}{4} \le t \le \frac{s+2}{4} \\ \gamma\big(\frac{4t-s-2}{2-s}\big) & \text{if } \frac{s+2}{4} \le t \le 1 \end{cases}$$

$$h_2(s,t) = \begin{cases} x_0 & \text{if } 0 \le t \le \frac{s}{2} \\ \alpha\big(\frac{2t-s}{2-s}\big) & \text{if } \frac{s}{2} \le t \le 1 \end{cases}$$

$$h_3(s,t) = \begin{cases} \alpha\big(\frac{2t}{2-s}\big) & \text{if } 0 \le t \le \frac{2-s}{2} \\ x_0 & \text{if } \frac{2-s}{2} \le t \le 1 \end{cases}$$

$$h_4(s,t) = \begin{cases} \alpha\big(2t\big) & \text{if } 0 \le t \le \frac{s}{2} \\ \alpha\big(s\big) & \text{if } \frac{s}{2} \le t \le \frac{2-s}{2} \\ \alpha\big(2-2t\big) & \text{if } \frac{2-s}{2} \le t \le 1 \end{cases}$$

$$h_5(s,t) = \begin{cases} \alpha\big(1-2t\big) & \text{if } 0 \le t \le \frac{s}{2} \\ \alpha\big(1-s\big) & \text{if } \frac{s}{2} \le t \le \frac{2-s}{2} \\ \alpha\big(2t-1\big) & \text{if } \frac{2-s}{2} \le t \le 1 \end{cases}$$

By computing that the different cases coincide on the overlap we see that these functions are well–defined and continuous. It is easy to see that $h_i(s,0) = h_i(s,1) = x_0$ for all $s \in I$. Then it only remains to compute that

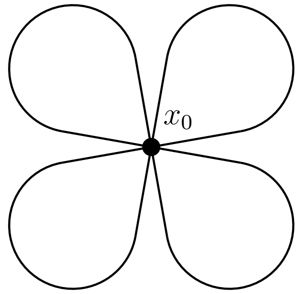
$$\begin{aligned} h_1(0,t) &= ((\alpha * \beta) * \gamma)(t), & h_1(1,t) &= (\alpha * (\beta * \gamma))(t), \\ h_2(0,t) &= \alpha(t), & h_2(1,t) &= (c * \alpha)(t), \\ h_3(0,t) &= \alpha(t), & h_3(1,t) &= (\alpha * c)(t), \\ h_4(0,t) &= \alpha(1) = c(t), & h_4(1,t) &= (\alpha * \overline{\alpha})(t), \\ h_5(0,t) &= \alpha(1) = c(t), & h_5(1,t) &= (\overline{\alpha} * \alpha)(t). \end{aligned}$$

Exercise 3. Let $x_0 \in S^1$ be a fixed point. Let

$$X = (S^1 \times \{1, \ldots, n\})/\!\sim$$

with the equivalence relation defined by $(x_0, i) \sim (x_0, j)$ for all i, j (the finite set $\{1, \ldots, n\}$ carries the discrete topology). It is called a *bouquet of circles* or *connected sum of circles*. Show that it is connected and Hausdorff, but not locally Euclidean if $n \ge 2$.

Hint: What happens if you remove a point?



Solution 3. Let us call the point the natural projection of this quotient p, and the point $p(x_0, i)$, which is the same for all i, just x.

To see that X is connected, we can for example show that there is a path from any point $y = p(y_0, i)$ to the special point $x \in X$. We know that S^1 is path–connected, so there is a path from (y_0, i) to (x_0, i). The projection of this path to X is a path from $y = p(y, i)$ to $x = p(x_0, i)$.

By Lemma 4.5 the Hausdorff property follows if we can show that $p\colon S^1 \times \{1, \ldots, n\} \to X$ is a closed map. Indeed, if $C \subset S^1 \times \{0, \ldots, n\}$ is closed, $p^{-1}(p(C))$ is either C or $C \cup \{(x_0, 1), \ldots, (x_0, n)\}$. In either case, this is closed.

Finally, let $y \in S^1 \setminus \{x_0\}$ and let $A = p(S^1 \setminus \{y\} \times \{1, \ldots, n\})$. If X was locally Euclidean, then A would also be locally Euclidean. A is connected, and $A \setminus \{x\}$ has $2n \geq 4$ connected components. This is a contradiction to the following Lemma.

Definition 1. A subset $A \subset \mathbb{R}^n$ is *convex* if for every two points $x, y \in A$ the line segment $\overline{xy}$ between them is contained in A.

A subset $A \subset \mathbb{R}^n$ is *star shaped* if there is a point $x_0 \in A$ such that for every point $x \in A$ the line segment $\overline{x_0 y}$ is contained in A.

Exercise 4.

a) Find a set which is star shaped, but not convex.

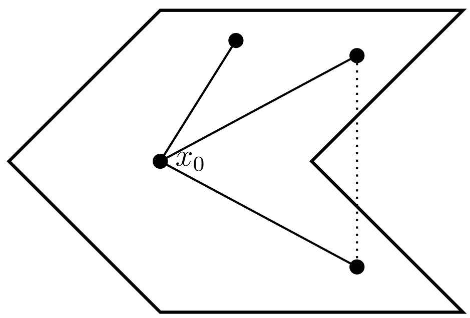
b) Show that $\pi_1(A, x_0)$ is the trivial group for every star shaped set $A \subset \mathbb{R}^n$.

Solution 4.

Lemma 1. If X is a connected locally Euclidean space and $x \in X$, then $X \setminus \{x\}$ has at most two connected components.

Proof. If $m = 0$, then X must be a single point and the statement is clearly true. So assume $m \geq 1$. Let $U \subset X$ be a neighborhood of x which is homeomorphic to a ball in $\mathbb{R}^m$, such that the homeomorphism sends x to the origin. Then $U \setminus \{x\}$ is either connected (if $m = 2$), or has two connected components U_1 and U_2 (if $m = 1$). Let $C \subset X \setminus \{x\}$ be a clopen subset which contains an element of $y \in U \cap \{x\}$ if $m \geq 2$ and which contains an element $y_1 \in U_1$ and $y_2 \in U_2$ each if $m = 1$. Then $C \cap U$ is clopen in $U \setminus \{x\}$ and non–empty, so $C \cap U = U \setminus \{x\}$ or $C \cap U = U_i$ for $i \in \{1, 2\}$. The latter is impossible because $y_1, y_2 \in C \cap U$. So $C \cap U = U \setminus \{x\}$. Then $C \cup U = C \cup \{x\}$ is clopen in X, so $C \cup \{x\} = X$, hence $C = X \setminus \{x\}$. This shows that the connected component containing y resp. the union of the connected components containing y_1 and y_2 is all of X. □

a)



b) The argument is essentially the same as for $\mathbb{R}^n$. If $\gamma \colon I \to A$ is a loop at $x_0 \in A$, then

$$h(s, t) = (1 - s)x_0 + s\gamma(t)$$

defines a continuous map, with $h(s, 0) = x_0$, $h(s, 1) = x_0$, $h(0, t) = x_0$, $h(1, t) = \gamma(t)$. To see that h is a homotopy from the constant curve to γ in A, we also need to know that $h(s, t) \in A$ for all s, t. This is where the star shaped property comes into play:

$$h(s, t) \in \overline{x_0\gamma(t)} \subset A.$$