Solutions 12

Exercise 1. Let X be a topological space and $\mathcal{U}$ an open cover of X. Let $\gamma\colon I \to X$ be a path. Show that there are real numbers $0 = t_0 < t_1 < \cdots < t_n = 1$ and $U_1, \ldots, U_n \in \mathcal{U}$ with

$$\gamma([t_{i-1}, t_i]) \in U_i \qquad \forall i \in \{1, \ldots, n\}.$$

Solution 1. The connected componenents of the sets $\gamma^{-1}(U)$ for all $U \in \mathcal{U}$ form an open cover of I. Since I is compact, there is a finite subcover. This means there are open intervals $(a_1, b_1), \ldots, (a_n, b_n) \subset \mathbb{R}$ covering I and $U_1, \ldots, U_n \in \mathcal{U}$ with $\gamma((a_i, b_i) \cap I) \subset U_i$. We can assume that none of these intervals is contained in another, as otherwise we could remove the smaller one and still cover I. Further assume they are ordered such that $a_1 < a_2 < \cdots < a_n$. Then we also have $b_1 < b_2 < \cdots < b_n$, as $b_{i+1} < b_i$ would imply $(a_{i+1}, b_{i+1}) \subset (a_i, b_i)$. We can also assume that all a_i and b_i except a_1 and b_n are in I. All the intervals $(a_1, b_1), \ldots, (a_i, b_i)$ are contained in $(-\infty, b_i)$ and all the intervals $(a_{i+1}, b_{i+1}), \ldots, (a_n, b_n)$ are in (a_{i+1}, ∞). Hence $b_i > a_{i+1}$ for all i, as else they could not be a cover of I. Now choose $t_i \in (a_{i+1}, b_i)$. Then $\gamma([t_{i-1}, t_i]) \subset \gamma((a_i, b_i)) \subset U_i$ as required.

Exercise 2. Assume $n \geq 2$. We consider the n–dimensional sphere

$$S^n = \Big\{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} x_i^2 = 1\Big\},$$

with the two special points $N = (1, 0, \ldots, 0)$ and $S = (-1, 0, \ldots, 0)$. Let $\gamma\colon I \to S^n$ be a loop with $\gamma(0) = \gamma(1) = N$.

a) Show that $S^n \setminus \{N\}$ and $S^n \setminus \{S\}$ are simply connected.

b) Show that there exists $k \in \mathbb{N}$ and real numbers $0 = t_0 < t_1 < \cdots < t_{2k+1} = 1$ such that, for all suitable $i \in \mathbb{N}$,

$$\gamma([t_{2i}, t_{2i+1}]) \in S^n \setminus \{S\}, \qquad \gamma([t_{2i-1}, t_{2i}]) \in S^n \setminus \{N\}.$$

That is, the sections alternatingly avoid either S or N.

c) Show that γ is homotopic to a loop which avoids S.

d) Show that γ is homotopic to a constant loop, and hence S^n is simply connected.

Solution 2.

a) Both of these are homeomorphic to $\mathbb{R}^n$ using the stereographic projections

$$S^n \setminus \{N\} \to \mathbb{R}^n, \quad (x_0, \ldots, x_n) \mapsto \left(\frac{x_1}{1 - x_0}, \ldots, \frac{x_n}{1 - x_0} \right),$$
$$S^n \setminus \{S\} \to \mathbb{R}^n, \quad (x_0, \ldots, x_n) \mapsto \left(\frac{x_1}{1 + x_0}, \ldots, \frac{x_n}{1 + x_0} \right).$$

b) By exercise 1 there are $0 = t'_0 < t'_1 < \cdots < t'_n = 1$ such that every $\gamma([t_{i-1}, t_i])$ is either contained in $S^n \setminus \{N\}$ or $S^n \setminus \{S\}$. By combining some of these segments, we can make them alternate between $S^n \setminus \{S\}$ and $S^n \setminus \{N\}$. Since the start point and end point of γ is N, the resulting number of segments must be odd and both start and end with one which is mapped into $S^n \setminus \{S\}$.

c) Decompose the loop as in b). Each even segment $\gamma|_{[t_{2i-1}, t_{2i}]}$ is in $S^n \setminus \{N\}$, and the endpoints $\gamma(t_{2i-1})$ and $\gamma(t_{2i})$ also can't be S. Since $S^n \setminus \{N\}$ is simply connected, $\gamma|_{[t_{2i-1}, t_{2i}]}$ is homotopic to any other path between these two endpoints, in particular one which does not cross S (this just uses that $S^n \setminus \{N, S\} \cong \mathbb{R}^n \setminus \{0\}$ is path connected). The odd also avoid S, so applying a homotopy like this to every even segment gives us a homotopy from γ to a loop avoiding S.

d) By c) γ is homotopic to a loop in $S^n \setminus \{S\}$, and as $S^n \setminus \{S\}$ is simply connected, this is homotopic to a constant loop.

Exercise 3. Let X be a topological space, $A \subset X$ and $x_0 \in A$. A continuous map

$$h \colon I \times X \to X$$

such that

- $h(0, x) = x$ for all $x \in X$,
- $h(1, a) = a$ for all $a \in A$,
- $h(1, x) \in A$ for all $x \in X$

is called a *deformation retraction* of X onto A.

Show that if X admits a deformation retraction onto A, then $\pi_1(X, x_0) \cong \pi_1(A, x_0)$. Use this to show that $\pi_1(\mathbb{R}^2 \setminus \{0\}) \cong \pi_1(S^1)$.

Solution 3. Let $\iota\colon A \to X$ be the inclusion map and $\iota_*\colon \pi_1(A, x_0) \to \pi_1(X, x_0)$ the induced homomorphism on fundamental groups. We need to show that ι_* is bijective.

For injectivity, let α, β be loops in A at x_0 which are homotopic in X, via a homotopy H. Then

$$H'\colon I \times I \to A, \qquad H'(s,t) = h(1, H(s,t))$$

is a homotopy in A from α to β, so $[\alpha] = [\beta]$ in $\pi_1(A, x_0)$.

For surjectivity, let γ be a loop in X at x_0 and define the loop γ' in A by $\gamma'(t) = h(1, \gamma(t))$. We would like to say that γ' and γ are homotopic in X by the homotopy

$$H_1\colon I \times I \to X, \qquad H_1(s,t) = h(s, \gamma(t)).$$

And indeed $H_1(0,t) = \gamma(t)$ and $H_1(1,t) = \gamma'(t)$, but H_1 is not a homotopy of paths because $H_1(s,0)$ and $H_1(s,1)$ need not be constant. They would be constant if we had required $h(t,a) = a$ for all $t \in I, a \in A$ in the definition of deformation retractions.

To fix this, we could for example define paths σ, τ in X by $\sigma(t) = h(t, x_0)$ and $\tau(t) = h(1, \sigma(t))$, and another "homotopy" by

$$H_2\colon I \times I \to X, \qquad H_2(s,t) = \begin{cases} h(1-2t, x_0) & \text{if } t \le \frac{1-s}{2} \\ h(s, \sigma(\frac{2t+s-1}{s+1})) & \text{if } t \ge \frac{1-s}{2}. \end{cases}$$

It is continuous and satisfies

$$\begin{aligned} H_2(0,t) &= (\overline{\sigma} * \sigma)(t), \\ H_2(1,t) &= \tau(t), \\ H_2(s,0) &= h(1, x_0) = x_0, \\ H_2(s,1) &= h(s, \sigma(1)) = h(s, x_0) = H_1(s,0) = H_1(s,1). \end{aligned}$$

So we can piece these together to get

$$H_3\colon I \times I \to X, \qquad H_3(s,t) = \begin{cases} H_2(s, 3t) & \text{if } t \le \frac{1}{3} \\ H_1(s, 3t-1) & \text{if } \frac{1}{3} \le t \le \frac{2}{3} \\ H_2(s, 3-3t) & \text{if } t \ge \frac{2}{3} \end{cases}$$

which is an *actual* homotopy from $(\overline{\sigma} * \sigma) * \gamma * (\overline{\sigma} * \sigma)$ to $\tau * \gamma' * \overline{\tau}$. This shows that

$$\gamma \simeq \overline{\sigma} * \sigma * \gamma * \overline{\sigma} * \sigma \simeq \tau * \gamma' * \overline{\tau}.$$

But $\tau * \gamma' * \overline{\tau}$ is a curve in A, hence we have $\iota_*([\tau * \gamma' * \overline{\tau}]) = [\gamma]$.

Finally, it is easy to check that

$$h\colon I \times (\mathbb{R}^2 \setminus \{0\}) \to \mathbb{R}^2 \setminus \{0\}, \qquad (s,x) \mapsto \frac{s + (1-s)\|x\|}{\|x\|}\, x$$

is a deformation retraction of $\mathbb{R}^2 \setminus \{0\}$ onto S^1.

Exercise 4. Let X, Y be topological spaces, $x_0 \in X$, $y_0 \in Y$, and

$$p_1\colon X \times Y \to X, \qquad p_2\colon X \times Y \to Y$$

the projections. Show that the map

$$p_{1*} \times p_{2*}\colon \pi_1(X \times Y, (x_0, y_0)) \to \pi_1(X, x_0) \times \pi_1(Y, y_0)$$

is a group isomorphism, so $\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$.

Solution 4. We already know that this map is a homomorphism of groups, so we only need to show that it is injective and surjective.

For injectivity, let $\alpha, \beta\colon I \to X \times Y$ be two loops at (x_0, y_0) and $\alpha_i = p_i \circ \alpha$, $\beta_i = p_i \circ \beta$ their projections. Assume that $\alpha_1 \simeq \beta_1$ and $\alpha_2 \simeq \beta_2$ via homotopies $h_1\colon I \times I \to X$ and $h_2\colon I \times I \to Y$. Then $h\colon I \times I \to X \times Y$ defined by $h(s,t) = (h_1(s,t), h_2(s,t))$ is a homotopy from α to β.

For surjectivity, let γ_1 be a loop in X at x_0 and γ_2 a loop in Y at y_0. Then γ defined by $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ is a loop in $X \times Y$ with $p_{1*}([\gamma]) = [\gamma_1]$ and $p_{2*}([\gamma]) = [\gamma_2]$.