Fall 2020 – Math 408D – 53185/53190/53225/53230

Extra problems

some voluntary, maybe slightly harder problems if you think the Quest problems are too easy and would like to have some extra practice. The exam will be closer to the usual homeworks or the problems in the book.

Exercise 1. Find

$$\lim_{n\to\infty} \sqrt[n]{n}.$$

Exercise 2. Find

$$\lim_{n\to\infty} \sqrt{n+\sqrt{n}} - \sqrt{n}.$$

Exercise 3. For any positive numbers a, b, find

$$\lim_{n\to\infty} \frac{a^{n+1}+b^{n+1}}{a^n+b^n}.$$

Exercise 4.

a) Convince yourself (for example by drawing a sketch of the graph) that

$$\int_1^n \ln x \,\mathrm{d}x \le \sum_{k=1}^{n} \ln k \le \ln n + \int_2^n \ln x \,\mathrm{d}x$$

b) Using a), prove the inequalities (a version of the *Stirling approximation*)

$$1 \le \ln n! - n\ln n + n \le \ln n + 2 - 2\ln 2.$$

c) Find

$$\lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n}.$$

Exercise 5. Find the Taylor series of

$$f(x) = \frac{1}{x^2-1}$$

at a (Hint: a partial fraction decomposition might help). What is its radius of convergence?

Exercise 6. Consider the function

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

a) Show that the n–th derivative of f is of the form

$$f^{(n)}(x) = p_n\left(\frac{1}{x}\right) e^{-1/x^2}$$

for every $x \neq 0$, where p_n is some polynomial.

b) Using a), show that f is n–times differentiable at 0 and $f^{(n)}(0) = 0$.

Exercise 7. The goal of this exercise is to prove the integral form for the n–th remainder of the Taylor polynomial of a function f at a:

$$R_n(x) = \int_a^x \frac{f^{(n+1)}(t)}{n!}(x-t)^n \, \mathrm{d}t.$$

a) Show the formula in the the case $n = 0$, that is

$$R_0(x) = \int_a^x f'(t) \, \mathrm{d}t.$$

b) Show the formula for $n = 1$, that is

$$R_1(x) = \int_a^x f''(t)(x-t) \, \mathrm{d}t$$

using integration by parts.

c) For the general case, use integration by parts to show that

$$\int_a^x \frac{f^{(n+1)}(t)}{n!}(x-t)^n \, \mathrm{d}t + T_n(x) = \int_a^x \frac{f^{(n)}(t)}{(n-1)!}(x-t)^{n-1} \, \mathrm{d}t + T_{n-1}(x).$$

Deduce from this that

$$\int_a^x \frac{f^{(n+1)}(t)}{n!}(x-t)^n \, \mathrm{d}t + T_n(x) = f(x),$$

proving the integral formula for $R_n(x)$.