COMPLEX ANALYSIS – HOMEWORK ASSIGNMENT 12

Due Friday, May 3, 2013, at the beginning of class.

Please write clearly, and staple your work !

1. Problem

Consider the Poisson kernel (we are dropping the tilde from the notation used in class)

$$P_r(\theta) = \frac{1-r^2}{1-2r\cos(2\pi\theta)+r^2} \quad , \quad 0<r<1 \, , \; \theta \in (-\frac{1}{2}, \frac{1}{2}] \, .$$

Prove that the following hold:

(1) $\int_{-\frac{1}{2}}^{\frac{1}{2}} P_r(\theta)d\theta = 1$ for all $0<r<1$.

(2) $\sup_{0<r<1} \int_{-\frac{1}{2}}^{\frac{1}{2}} |P_r(\theta)|d\theta < \infty$.

(3) For any $\eta > 0$,

$$\lim_{r\to 1^-} \int_{|\theta|>\eta} |P_r(\theta)|d\theta = 0$$

(4) $P_r(\theta) = P_r(-\theta) = P_r(\theta+1)$ is a 1-periodic, even function, monotonically decreasing in $\theta \in (0, \frac{1}{2}]$, and can be written in the form

$$P_r(\theta) = \int_0^{\frac{1}{2}} \chi_{[-\phi,\phi]}(\theta) d\mu(\phi)$$

for $0<r<1$, where the measure μ is defined by

$$d\mu(\phi) = (\partial_\phi P_r(\phi))d\phi - P_r\Big(\frac{1}{2}\Big)\delta\Big(\phi - \frac{1}{2}\Big)d\phi \, ,$$

with δ denoting the Dirac distribution (defined by $\int g(x)\delta(x)dx = g(0)$).

2. Problem

Assume that $f \in C(\partial\mathbb{D})$ is a continuous function on $\partial\mathbb{D}$, parametrized by $\theta \in (-\frac{1}{2}, \frac{1}{2}]$ (thus, $f(\theta+1) = f(\theta)$). Prove that

$$u(z) = (P_r * f)(\theta) = \int_{-\frac{1}{2}}^{\frac{1}{2}} P_r(\theta-\phi)f(\phi)d\phi \quad , \quad z = re^{2\pi i\theta} \, ,$$

is harmonic in $\mathbb{D}$.

3. Problem

Let $f \in C(\partial\mathbb{D})$ be a continuous function on $\partial\mathbb{D}$, parametrized by $\theta \in (-\frac{1}{2}, \frac{1}{2}]$. Prove that

$$\lim_{r\to 1^-} \sup_{\theta\in(-\frac{1}{2},\frac{1}{2}]} \Big|(P_r * f)(\theta) - f(\theta)\Big| = 0 \, .$$