COMPLEX ANALYSIS – HOMEWORK ASSIGNMENT 4

Due Friday, February 15, 2013, at the beginning of class.

Please write clearly, and staple your work !

1. Problem

Find all automorphisms of $\mathbb{D}$, $\mathbb{H}$, and $\mathbb{C}$.

Hint: For an arbitrary automorphism $f : \mathbb{D} \to \mathbb{D}$, show that you can assume $f(0) = 0$ via composition with a fractional linear transformation. Then, apply the Schwarz lemma.

2. Problem

If f is analytic on $\Omega = \{z \in \mathbb{C} \,|\, Re(z) > 0 \text{ } or \text{ } Im(z) > 0\}$ and satisfies $|f'(z)| \leq 2$ for all $z \in \Omega$, show that $|f(z) - f(\omega)| \leq 3|z - \omega|$ for all $z, \omega \in \Omega$.
For the case of $f : \mathbb{C} \to \mathbb{C}$, assume that $f(0) = 0$, and consider the function $g(z) := \frac{1}{f(\frac{1}{z})}$.

3. Problem

Assume that $f : \mathbb{D} \to \mathbb{C}$ is holomorphic, with $Re(f(z)) > 0$ for all $z \in \mathbb{D}$. Moreover, assume that $f(0) = a > 0$. Prove that $|f'(0)| \leq 2a$. Does there exist such an f fulfilling equality, $|f'(0)| = 2a$?

Hint: Verify that the fractional linear transformation $T : z \mapsto \frac{z-a}{z+a}$ maps the right half plane to $\mathbb{D}$. Then, consider the function $g := T \circ f$, and use the Schwarz lemma.

4. Problem

Let $f : \mathbb{D} \to \mathbb{C}$ be a holomorphic function such that $Re(f(z)) > 0$ for all $z \in \mathbb{D}$. Prove that there exist bounded holomorphic functions $g, h : \mathbb{D} \to \mathbb{C}$ such that $f(z) = \frac{g(z)}{h(z)}$, $z \in \mathbb{D}$.

5. Problem

Assume that $\{z_j\}_{j=1}^{\infty} \subset \{z \,|\, Re(z) > 0\}$ is a given sequence. True or false: If $\sum_j z_j$ and $\sum_j z_j^2$ both converge, then $\sum_j |z_j|^2$ also converges.