

## COMPLEX ANALYSIS – HOMEWORK ASSIGNMENT 5

Due Friday, February 22, 2013, at the beginning of class.

**Please write clearly, and staple your work !**

### 1. PROBLEM

Assume that  $f \in \mathcal{H}(\Omega \setminus \{z_0\})$ , and that  $\lim_{z \rightarrow z_0} (z - z_0)f(z) = 0$ . Define  $g(z) := (z - z_0)f(z)$  for  $z \in \Omega \setminus \{z_0\}$ , and  $g(z_0) := 0$ . Prove that  $g$  is holomorphic in  $\Omega$ .

### 2. PROBLEM

Determine the Laurent series of the function  $f(z) = \frac{1}{1-z}$  relative to the point  $z_0 = 2$  in the region  $\{z \mid |z - 2| > \frac{3}{2}\}$ .

### 3. PROBLEM

Evaluate the integral

$$\int_0^{2\pi} \frac{dt}{3 + 2 \cos t}$$

by converting it into a contour integral along the unit circle  $\gamma(t) = e^{it}$ , and using that  $2 \cos t = z + 1/z$  for  $z = e^{it}$ .

### 4. PROBLEM

Assume that  $f$  is holomorphic in a region  $\Omega$  and satisfies the inequality  $|f(z) - 1| < 1$  for all  $z \in \Omega$ . Prove that

$$\oint_{\gamma} \frac{f'(z)}{f(z)} dz = 0$$

for every closed  $C^1$ -curve in  $\Omega$ .

### 5. PROBLEM

Let  $f$  be a meromorphic function in  $\mathbb{C}$  with finitely many poles, located at  $\{z_j\}_{j=1}^J$ . Prove that

$$\sum_{j=1}^J \operatorname{res}(f; z_j) = \operatorname{res}(g; 0)$$

where  $g(z) := \frac{1}{z^2} f\left(\frac{1}{z}\right)$ . Here,  $\operatorname{res}(f; z_j)$  denote the residue of  $f$  at  $z_j$ . It is defined by  $\operatorname{res}(f; z_j) = a_{-1}$  if  $f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_j)^n$  is the Laurent series of  $f$  at  $z_j$ .