COMPLEX ANALYSIS – HOMEWORK ASSIGNMENT 5

Due Monday, February 24, 2014, at the beginning of class.

Please write clearly, and staple your work !

1. Problem

Assume that $f \in \mathcal{H}(\Omega \setminus \{z_0\})$, and that $\lim_{z \to z_0}(z - z_0)f(z) = 0$. Define $g(z) := (z - z_0)f(z)$ for $z \in \Omega \setminus \{z_0\}$, and $g(z_0) := 0$. Prove that g is holomorphic in Ω.

2. Problem

Determine the Laurent series of the function $f(z) = \frac{1}{1-z}$ relative to the point $z_0 = 2$ in the region $\{z \,|\, |z - 2| > \frac{3}{2}\}$.

3. Problem

Evaluate the integral

$$\int_0^{2\pi} \frac{dt}{3 + 2\cos t}$$

by converting it into a contour integral along the unit circle $\gamma(t) = e^{it}$, and using that $2\cos t = z + 1/z$ for $z = e^{it}$.

4. Problem

Assume that f is holomorphic in a region Ω and satisfies the inequality $|f(z) - 1| < 1$ for all $z \in \Omega$. Prove that

$$\oint_\gamma \frac{f'(z)}{f(z)} dz = 0$$

for every closed C^1-curve in Ω.

5. Problem

Let f be a meromorphic function in $\mathbb{C}$ with finitely many poles, located at $\{z_j\}_{j=1}^J$. Prove that

$$\sum_{j=1}^J \operatorname{res}(f; z_j) = \operatorname{res}(g; 0)$$

where $g(z) := \frac{1}{z^2} f(\frac{1}{z})$. Here, $\operatorname{res}(f; z_j)$ denote the residue of f at z_j. It is defined by $\operatorname{res}(f; z_j) = a_{-1}$ if $f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_j)^n$ is the Laurent series of f at z_j.