COMPLEX ANALYSIS – HOMEWORK ASSIGNMENT 8

Due Monday, March 31, 2014, at the beginning of class.

Please write clearly, and staple your work !

1. Problem

Consider the Weierstrass factor of order m, $E_m(z) = (1-z)\exp(\sum_{\ell=1}^{m} \frac{z^\ell}{\ell})$.
(i) Prove that

$$|E_m(z) - 1| \leq |z|^{m+1} \quad \text{for all } z \in \mathbb{D}\,.$$

Hint: Consider the logarithmic derivative $\frac{d}{dz}\mathrm{Log}E_m(z)$.
(ii) Prove that

$$\ln(|E_m(z)|) \leq (2m+1)|z|^{m+1} \quad \text{for all } z \in \mathbb{C}\,.$$

2. Problem

Assume that $\{z_j\}_{j\in\mathbb{N}} \subset \mathbb{C}^*$ is a sequence with no accumulation points. Consider the infinite product

$$f(z) = \prod_{j=1}^{\infty} E_{m_j}(\frac{z}{z_j})\,.$$

(i) Prove that the product converges for all z, and defines an entire function for $m_j = j$.
(ii) Assume that $\sum_j \frac{1}{|z_j|^{p+1}} < \infty$. Prove that the product converges for all z, and defines an entire function for $m_j = p$ (independent of j).

3. Problem

Show that for any positive real number r,

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{(n!)^{1/r}}$$

defines an entire function f of order r.

4. Problem

Find all entire functions f that satisfy $f(\sqrt{n}) = n^2$ for every positive integer n, and $|f(z)| \leq e^{3|z|}$ for every complex number z.