PDE I – HOMEWORK ASSIGNMENT 10

Due Monday, November 15, 2010. **Please write clearly, and staple your work !**

1. Problem

Let

$$\|f\|_{L^p_{t,x}} := \Big(\int |f(t,x)|^p dt dx \Big)^{\frac{1}{p}} .$$

Define the maps

$$Tf := e^{it\Delta} f$$

and

$$T^* f := \int_0^t e^{-is\Delta} f(s,\cdot) ds \, .$$

Prove that

$$\|TT^* f\|_{L^{p'}_{t,x}} \leq C \, \|f\|_{L^p_{t,x}}$$

if $p' = 2 + \frac{4}{n}$.

Hint: First prove that

$$\|TT^* f\|_{L^{p'}_{x}} \leq \int_{\mathbb{R}} \Big(\frac{1}{|t-s|} \Big)^{d(\frac{1}{p}-\frac{1}{2})} \|f(s,\cdot)\|_{L^p_x} ds$$

using the result proved in the midterm. Then, use the Hausdorff-Sobolev-Young inequality in the variables t, s.