PDE I – HOMEWORK ASSIGNMENT 12

For Friday, December 3, 2010. **Please write clearly, and staple your work !**

1. Problem

Consider the Schrodinger operator

$$H = -\Delta + \eta V$$

where $\eta > 0$ is a constant.

- Assume that $V \in L^{\frac{3}{2}}(\mathbb{R}^3)$ and $V(x) \leq 0$ for all $x \in \mathbb{R}^3$. Using the Birman-Schwinger principle, prove that H has no bound states (negative eigenvalues) if $\eta > 0$ is sufficiently small.

- Assume now that $x \cdot \nabla V \leq 0$ for all $x \in \mathbb{R}^n$, and let $\eta = 1$. Prove that then, H has no bound states.
  Proceed as follows: Consider the operator
  $$A := -\frac{1}{2}(x \cdot (i\nabla) + (i\nabla) \cdot x).$$
  Prove that if ψ is an eigenfunction of H, then $(\psi, i[H,A]\psi)_{L^2} = 0$ (recall that $[H,A] = HA - AH$ is the commutator).

  Then, prove that $i[H,A] = -\Delta - x \cdot \nabla V$, and conclude the argument.

- Assume that $I \subset \mathbb{R}$ is an interval intersecting the spectrum of H. Let $P(I)$ denote the projection-valued measure of I associated to H, and assume that there exists a constant $\theta > 0$ such that
  $$P(I)\, i[H,A]\, P(I) \geq \theta P(I)$$
  in the sense of positive operators.
  Prove that the interval I contains no point spectrum of H.
  (This is known as the Mourre estimate).

2. Problem

Reading assignment: Read chapter 7.1 (beginning) and 7.1.2 of Evans' book on the existence and uniqueness of weak solutions of parabolic problems using Galerkin approximations.