PDE I – HOMEWORK ASSIGNMENT 2

Due Monday, September 13, 2010. **Please write clearly, and staple your work !**

1. Problem

Consider the 1-dimensional heat equation $(\partial_t - \partial_{xx})u(x,t) = 0$ for $t, x \in \mathbb{R}$. Assume boundary conditions $u(t,0) = g(t)$, and $\partial_x u(0,t) = 0$ at $x = 0$, for $t \in \mathbb{R}$.

- Use the ansatz
$$u(x,t) = \sum_{n=0}^{\infty} h_n(t) x^{2n}$$
to recursively determine $(h_n(t))_{n\in\mathbb{N}_0}$. Why do no odd powers in x occur here?
- Let $\alpha > 1$, and
$$g(t) := \begin{cases} 0 & \text{for } t \leq 0 \\ \exp(-t^{-\alpha}) & \text{for } t > 0\,. \end{cases}$$
Show that in this case, the sequence of coefficients $(h_n(t))_n$ satisfies
$$|h_n(t)| < \frac{n!}{(2n)!\,(C_0 t)^n} \exp(-\frac{1}{2}t^{-\alpha})$$
for some constant C_0.
- Conclude that the series expansion of $u(x,t)$ converges for every $x \in \mathbb{R}$, and every $t > 0$, and that
$$|u(x,t)| \leq C_1 \exp\big(C_0^{-1} t^{-1} x^2 - \frac{1}{2} t^{-\alpha} \big)$$
holds, for some constant $C_1 > 0$.
- Observe that for any $\alpha > 1$, we have obtained a solution of the heat equation (due to Tychonoff) which is zero at $t = 0$, but which is nonzero for $t > 0$. How can you explain this result in view of the uniqueness result for solutions of the heat equation proven in class (where zero initial data implies that the solution is zero; Thm 7 in Chapter 2.3 of Evans) ?

2. Problem

Let $U \subset \mathbb{R}^n$ be bounded and open, and consider the parabolic cylinder $U_T := U \times (0,T]$ for $T > 0$. A function $v \in C_1^2(U_T)$ is a subsolution of the heat equation if
$$(\partial_t - \Delta)v \leq 0 \;\text{ in } U_T\,.$$

- Prove that for any subsolution v,
$$v(x,t) \leq \frac{1}{4r^n} \int\int_{E(x,t;r)} v(y,s) \frac{|x-y|^2}{(t-s)^2}\, ds\, dy$$
for any heat ball $E(x,t;r) \subset U_T$.
- Prove that therefore, $\max_{\overline{U_T}} v = \max_{\Gamma_T} v$, where $\Gamma_T = \overline{U_T} \setminus U_T$.